A VERY AMPLE LATTICE POLYTOPE WITH A NON-UNIMODAL h^*-VECTOR

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Abstract. Lattice polytopes are called *very ample* if for every sufficiently large k every lattice point of height k in the cone over the lattice polytope is the sum of k lattice points of height 1. This is a weakening of the well-known integer decomposition property (also called IDP). We give an example of a very ample lattice polytope whose h^*-vector is non-unimodal. Here, the h^*-vector is the coefficient vector of the numerator of the Ehrhart series of the lattice polytope. This answers a question of Ferroni and Higashitani, as well as a related question by Balletti. The main problem whether IDP lattice polytopes have unimodal h^*-vector is still unsolved. The example was found using ChatGPT 5.6 Sol. It is the Cartesian square of a lattice polytope belonging to a class of very ample examples constructed by Lasoń and Michałek.

1. Introduction and main result

Let us recall the main definitions[1]. We refer to [2] for references. A *lattice polytope* $P \subset \mathbb{R}^n$ is the convex hull of finitely many lattice points, i.e., elements of $\mathbb{Z}^n$. The *Ehrhart polynomial* of P is the unique polynomial $E_P(t)$ with $E_P(k) = |kP \cap \mathbb{Z}^n|$ for $k \in \mathbb{N}_{\geq 1}$. The h^*-*polynomial* $h^*_P(t)$ of P is defined as the numerator of its *Ehrhart generating series*:

$$\sum_{k=0}^{\infty} E_P(k)t^k = \frac{h^*_P(t)}{(1-t)^{d+1}},$$

where d denotes the dimension of P. For $h^*_P(t) = \sum_{i=0}^{d} h^*_i t^i$, we call $(h^*_0, \ldots, h^*_d)$ the h^*-*vector* of P.

The following properties of lattice polytopes have been intensively studied over the previous years. We refer to the excellent survey papers [3] and [4]. An *IDP polytope* (an abbreviation for lattice polytopes with the *Integer Decomposition Property*) is a lattice polytope P such that for every $k \in \mathbb{N}_{\geq 2}$ each lattice point in kP is a sum of k lattice points in P. If there exists some positive integer N such that the previous property holds for all $k \geq N$, then P is called *very ample*. We say the h^*-vector of P is *unimodal* if $h^*_0 \leq h^*_1 \leq \cdots \leq h^*_j \geq h^*_{j+1} \geq \cdots \geq h^*_d$ for some $j \in \{0, \ldots, d\}$. It is called *log-concave* if $h^*_{i-1}h^*_{i+1} \leq (h^*_i)^2$ for every $i \in \{1, \ldots, s-1\}$ where $s \in \mathbb{Z}_{\geq 0}$ denotes the degree of the polynomial $h^*_P(t)$.

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[1] Much of this introduction is directly taken from [5]. We plan to merge both papers into one publication.

The following conjecture is currently considered as the main open question in Ehrhart theory. It is explicitly stated in [4, Conjecture 1.1] and [7, Question 1.1].

Conjecture 1.1. Every IDP polytope has a unimodal h^*-vector.

In [5] the authors used machine learning to find an example of an IDP polytope with non-log-concave h^*-vector showing that one cannot strengthen the implication in the conjecture. In this paper we used ChatGPT 5.6 Sol to find an example that shows that one also cannot weaken the assumption to very ampleness. This completely answers Question 3.9 by Ferroni and Higashitani in [4].

Let us now describe the example. It is the Cartesian square of a lattice polytope constructed by Lasoń and Michałek in [6] to disprove several conjectures on very ample but non-IDP lattice polytopes.

For this, let $e_1, \dots, e_{30}$ be the standard basis vectors of $\mathbb{R}^{30}$. We define for $i = 1, \dots, 30$

$$u_i \coloneqq e_i + e_{i+1} \in \mathbb{Z}^{30}$$

with indices cyclic modulo 30, and set

$$Q := \operatorname{conv}((u_1, 332), (u_1, 333), (u_i, 0), (u_i, 1) : 2 \leq i \leq 30) \subset \mathbb{R}^{30} \times \mathbb{R}.$$

In the notation of [6], this is precisely $\mathcal{P}_{15,332}$, a lattice segmental fibration over the edge polytope of the even cycle with 30 vertices.

Proposition 1.2. The lattice polytope $Q \subset \mathbb{R}^{31}$ has the following properties:

(1) its dimension is 29 and it has 60 vertices,
(2) it is very ample but not IDP,
(3) its h^*-polynomial is

$$h^*_Q(t) = 1 + 30(t + \cdots + t^{14}) + 346t^{15}.$$

Proof. Dimension, number of vertices and the h^*-polynomial can be easily computed with any suitable software package. Very ampleness of Q has been shown in [6] (more precisely, Theorem 3 and Proposition 6). It is not IDP by Theorem 12 in [6]. □

Now, as in Balletti's paper [1] taking a Cartesian product of Q with itself does the trick.

Theorem 1.3. $Q \times Q$ is a 58-dimensional very ample, non-IDP lattice polytope with 3600 vertices and non-unimodal h^*-vector.

Proof. Clearly, Cartesian products preserve very ampleness as well as being non-IDP. Note that for all nonnegative integers k we have $E_{Q\times Q}(k) = (E_Q(k))^2$. Therefore, for integers r we get for the rth-coefficient of $h^*_{Q\times Q}(t)$

$$h^*_r = \sum_{m=0}^{r} (-1)^{r-m} \binom{59}{r-m} E_Q(m)^2,$$

where by Proposition 1.2

$$E_Q(m) = \binom{m+29}{29} + \left(\sum_{i=1}^{14} 30\binom{m+29-i}{29}\right) + 346\binom{m+29-15}{29}.$$

Therefore, we can verify non-unimodality by evaluating

$$\begin{aligned} h^*_{25} &= 16313471086060598699 73, \\ h^*_{26} &= 16279404802532288125 68, \\ h^*_{27} &= 16281678303220842911 28. \end{aligned}$$ □

$Q = \mathcal{P}_{15,332}$ produces the minimal example given by the above construction in terms of dimension and volume. Unimodality also fails when we consider $Q = \mathcal{P}_{15,a}$ for $a \in \{332, \ldots, 338\}$.

As this is also an example of two very ample lattice polytopes such that their Cartesian product does not have a unimodal h^*-vector, this also answers Question 5.1(b) by [1] and strengthens his main result (Theorem 4.4). Note the similarity of the h^*-vectors of Q with the ones found by Balletti using genetic algorithms.

Remark 1.4. It may be worth mentioning that this construction can't be straightforwardly generalised to IDP lattice polytopes. Indeed, by [8, Theorem 3.2 (a)] the h^*-vector $(h_0, h_1, \ldots, h_s)$ (where s denotes the degree of the h^*-polynomial) of an IDP lattice polytope satisfies $h_1 = h_2 = \cdots = h_{s-1}$ and $h_s \leq h_1$ provided $h_s \geq 2$, $s \geq 3$, and $h_1 = h_i$ for some $2 \leq i \leq s-1$. However, it is necessary that h_s is much bigger than h_1 for the square of the lattice polytope to have a non-unimodal h^*-vector.

That said, TODO: maybe a slow increase at the beginning and then a large final jump could be still possible? Investigate further and provide some suggestions.

Question 1.5. Is it possible to find a d-dimensional very ample polytope whose h^*-vector is not-unimodal, and there is an index $i > d/2$ where the unimodality is violated?

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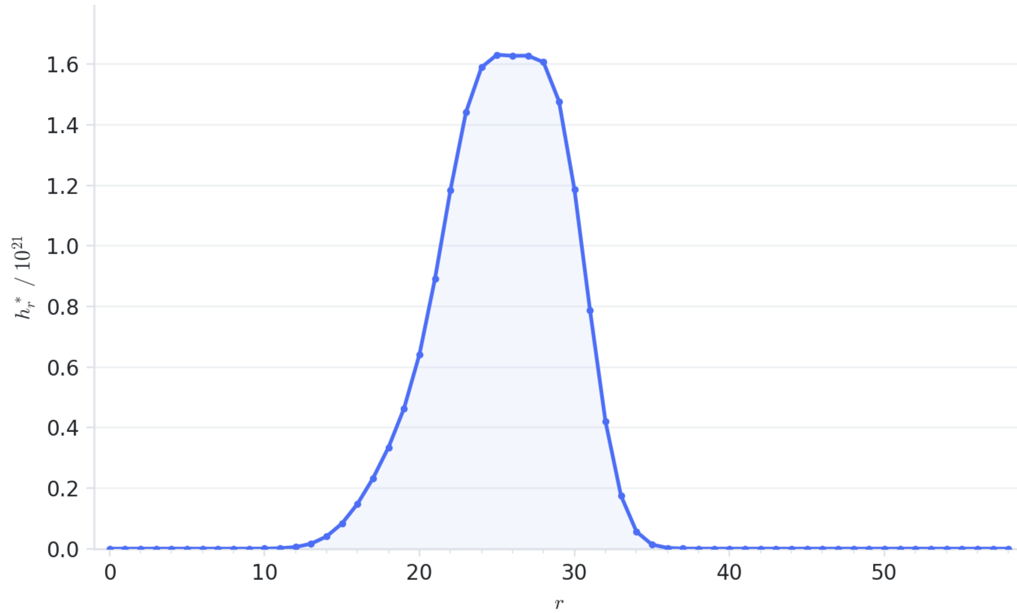

FIGURE 1. h^*-vector from Theorem 1.3

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