

The Euler Stratification for $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^n$

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Abstract

We study the Euler characteristic of a hypersurface in $(\mathbb{C}^*)^2 \times (\mathbb{C}^*)^n$ defined by a polynomial whose monomial support corresponds to lattice points in $\Delta_1 \times \Delta_1 \times \Delta_n$ as the coefficients of the defining polynomial vary. Each member of this hypersurface family corresponds to a three-way independence model from algebraic statistics, and the (signed) Euler characteristic is equal to the maximum likelihood degree (ML degree) of the model. We show in the case of $\Delta_1 \times \Delta_1 \times \Delta_1$ this Euler characteristic depends only on the vanishing patterns of the factors of the principal A -determinant, but this fails for $\Delta_1 \times \Delta_1 \times \Delta_n$ with $n \geq 2$. We prove that, for all $n \geq 1$, all positive integers up to the maximum possible ML degree can be realized as the Euler characteristic. Furthermore, we completely determine the Euler stratification for $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and provide partial information for $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$.

1 Introduction

In this paper, we consider the hypersurface $Y_{W,n} \subset (\mathbb{C}^*)^{2+n}$ defined by

$$f_W = f_0 + \sum_{k=1}^n z_k f_k = (w_{000} + w_{100}x_1 + w_{010}y_1 + w_{110}x_1y_1) + \sum_{k=1}^n z_k (w_{00k} + w_{10k}x_1 + w_{01k}y_1 + w_{11k}x_1y_1) \quad (1)$$

where $W = (w_{ijk}) \in (\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^{n+1})^* = (\mathbb{C}^*)^{4n+4}$. Our goal is to understand the variation of the Euler characteristic $\chi(Y_{W,n})$ with respect to W . The hypersurface $Y_{W,n}$ goes with the *scaled toric variety* $X_{W,n} \subset \mathbb{P}^{4n+3}$ defined as the scaled Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^n$

$$[x_0 : x_1] \times [y_0 : y_1] \times [z_0 : \cdots : z_n] \longrightarrow [w_{ijk}x_iy_jz_k : i, j = 0, 1, k = 0, 1, \dots, n] \quad (2)$$

into \mathbb{P}^{4n+3} ; see [ABB⁺19, Section 2.1]. The usual Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^n$ is obtained with $w_{ijk} = 1$ for all i, j, k , and we denote this toric variety by \mathcal{X}_n . The exponent vectors of the monomials in (2) are the lattice points of $P = \Delta_1 \times \Delta_1 \times \Delta_n$ where Δ_r is the convex hull of the standard unit vectors in \mathbb{R}^{r+1} . We collect these lattice points in the $(n+5) \times (4n+4)$ matrix A .

The *maximum likelihood estimation problem* is an optimization problem over the positive real part of $X_{W,n}$:

$$\text{maximize } \frac{\prod_{i,j,k} p_{ijk}^{u_{ijk}}}{\sum_{i,j,k} p_{ijk}} \quad \text{subject to } (p_{ijk}) \in X_{W,n} \cap \mathbb{R}_+^{4n+4}. \quad (3)$$

Here $(u_{ijk}) \in \mathbb{N}^{4n+4}$ is the data, and the variety $X_{W,n}$ is a *three-way independence model* corresponding to two binary and one $(n+1)$ -ary random variables that are probabilistically independent [Sul18, Chapter 9]. The *maximum likelihood degree* of $X_{W,n}$, $\text{mldeg}(X_{W,n})$, is defined to be the number of complex critical points of the rational objective function in (3) over $X_{W,n}$ for generic data; see [CHKS06, HKS05, HS14].

Our starting point is based on the following two results. The first one is a corollary of [Huh13, Theorem 1] or [HS14, Theorem 1.7]. For this let

$$\mathcal{H} = \left\{ (p_{ijk}) \in \mathbb{P}^{4n+3} : \left(\prod_{i,j,k} p_{ijk} \right) \left(\sum_{i,j,k} p_{ijk} \right) = 0 \right\}.$$

Proposition 1.1. *The ML degree of $X_{W,n}$ is equal to the signed Euler characteristic of $Y_{W,n}$:*

$$\text{mldeg}(X_{W,n}) = (-1)^{n+2} \chi(X_{W,n} \setminus \mathcal{H}) = (-1)^{n+1} \chi(Y_{W,n}).$$

We note that the second equality follows from the fact that $X_{W,n} \setminus \mathcal{H}$ and $(\mathbb{C}^*)^{n+2} \setminus Y_{W,n}$ are isomorphic essentially via the map in (2) and $\chi(Y_{W,n}) = -\chi((\mathbb{C}^*)^{n+2} \setminus Y_{W,n})$; see [TW24, Section 6.4] for further details. The second result is Theorem 13 of [ABB⁺19] applied to our case.

Proposition 1.2. *The ML degree of $X_{W,n}$ is at most the degree of the toric variety \mathcal{X}_n . Moreover, $\text{mldeg}(X_{W,n}) < \deg(\mathcal{X}_n)$ if and only if $E_A(W) = 0$ where E_A is the principal A -determinant.*

The principal A -determinant above [GKZ94, Chapter 10, Theorem 1.2] is a polynomial in the coefficients w_{ijk} with a known factorization

$$E_A = \prod_{\Gamma \text{ a face of } P} \text{Disc}(\Gamma \cap \mathbb{Z}^n),$$

where $\text{Disc}(\Gamma \cap \mathbb{Z}^n)$ is the A' -discriminant [GKZ94, Chapter 9]. Here, A' is the matrix with columns corresponding to vertices in the face Γ . The factors of E_A in the case of our toric variety \mathcal{X}_n are known explicitly, see [CHKO24, Section 2.4]. We will give more details about the principal A -determinant and its discriminantal factors in Section 2.

The Euler characteristic $\chi(Y_{W,n})$ and hence the ML degree $\text{mldeg}(X_{W,n})$ is controlled by the *Euler stratification* for $Y_{W,n}$ (or, by abuse of language, for $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^n$) [TW24]: the coefficient space where W lives can be stratified according to $\chi(Y_{W,n})$. A fundamental question about this stratification is summarized in the following.

Conjecture 1.3. [CHKO24, Conjecture 2.29] *If exactly the same factors of the principal A -determinant E_A vanish on W and W' then $\chi(Y_{W,n}) = \chi(Y_{W',n})$.*

This conjecture has a positive answer for the Segre product $\mathbb{P}^m \times \mathbb{P}^n$.

Theorem 1.4. [CHKO24, Theorem 1.3] *Let $Z_{W,m,n} \subset (\mathbb{C}^*)^m \times (\mathbb{C}^*)^n$ be the hypersurface defined by*

$$g = g_0 + \sum_{j=1}^n y_j g_j = \left(w_{00} + \sum_{i=1}^m w_{i0} x_i \right) + \sum_{j=1}^n y_j \left(w_{0j} + \sum_{i=1}^m w_{ij} x_i \right)$$

where $W = (w_{ij}) \in (\mathbb{C}^{m+1} \otimes \mathbb{C}^{n+1})^* = (\mathbb{C}^*)^{mn+m+n+1}$. Then $\chi(Z_{W,m,n}) = \chi(Z_{W',m,n})$ if exactly the same factors of the principal A -determinant E_A vanish on W and W' where A consists of the lattice points in $\Delta_m \times \Delta_n$. Equivalently, if the matroids defined by the matrices $[I_{m+1} W]$ and $[I_{m+1} W']$ are isomorphic, then $\chi(Z_{W,m,n}) = \chi(Z_{W',m,n})$.

Remark 1.5. Conjecture 1.3 is false for other families of scaled toric varieties. For instance, let

$$f = w_0 + w_1x + w_2x^2 + w_3x^3$$

where $W = (w_0, w_1, w_2, w_3) \in (\mathbb{C}^*)^4$. The corresponding toric variety is the twisted cubic in \mathbb{P}^3 and we let Z_W be the hypersurface in \mathbb{C}^* defined by f . The principal A -determinant E_A where

$$A = \begin{pmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

has a single relevant factor, namely, the classical discriminant D_f of the cubic polynomial f . The Euler characteristic $\chi(Z_W) = 3$ if and only if $D_f(W) \neq 0$. However, $D_f(W) = D_f(W') = 0$ does not necessarily imply $\chi(Z_W) = \chi(Z_{W'})$: The discriminant vanishes if and only if $f_W(x)$ is singular in which case $\chi(Z_W) < 3$, but the Euler characteristic is equal to the number of distinct roots of $f_W(x)$, i.e. it can be equal to 1 or 2.

1.1 Results

Our first result settles Conjecture 1.3 for $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^n$. In general, the statement that the vanishing of certain factors of the principal A -determinant determines $\chi(Y_{W,n})$ and with it the ML degree of $X_{W,n}$ is false. We provide a counterexample in Example 3.16. This also settles the question for general independence models. For $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^n$, one can however say the following:

Theorem 1.6. *Let f_W be as in (1) and let $Y_{W,n}$ be the corresponding hypersurface in $(\mathbb{C}^*)^{n+2}$.*

1. $n = 1$: *If exactly the same factors of the principal A -determinant E_A vanish on W and W' , then $\chi(Y_{W,n}) = \chi(Y_{W',n})$.*
2. $n = 2$: *The same statement holds as long as the factor of E_A corresponding to the $2 \times 2 \times 3$ hyperdeterminant does not vanish or as long as one of the factors corresponding to a $2 \times 2 \times 2$ hyperdeterminant does vanish.*
3. $n > 2$: *The same statement holds as long as the factors of E_A corresponding to $2 \times 2 \times 3$ hyperdeterminants do not vanish.*

We will prove this theorem by first computing $\text{mdeg}(X_{W,n})$ as in the proof of Theorem 1.4 from [CHKO24]. In particular, we will analyze the Euler characteristic of the possible intersections of the plane quadrics defined by f_0, f_1, \dots, f_n in (1).

Despite this theorem, it is difficult to compute the Euler stratification of $Y_{W,n}$ itself. Nevertheless we will determine the complete Euler stratification for $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. As we will state later, the principal A -determinant for $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ consists of seven factors one of which is a polynomial of degree 4 called the $2 \times 2 \times 2$ hyperdeterminant.

Theorem 1.7. *The Euler stratification for $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ consists of 41 strata:*

0. One stratum where no factor of E_A vanishes with $\chi(Y_{W,1}) = 6$ on this stratum.
1. 7 strata where exactly one factor of E_A vanishes with $\chi(Y_{W,1}) = 5$.
2. $\binom{7}{2}$ strata where exactly two factors of E_A vanish with $\chi(Y_{W,1}) = 4$.
3. 8 strata where three certain factors not containing the $2 \times 2 \times 2$ hyperdeterminant vanish with $\chi(Y_{W,1}) = 3$.
4. 3 strata where four certain factors and the $2 \times 2 \times 2$ hyperdeterminant vanish with $\chi(Y_{W,1}) = 2$.
5. One stratum where all factors of E_A vanish with $\chi(Y_{W,1}) = 1$.

Finally, we will turn to the question of the realizability of the ML degrees. We know that $\text{mldeg}(X_{W,n}) \leq \deg(\mathcal{X}_n)$ and with the choice of $w_{ijk} = 1$ for all $i, j = 0, 1$ and $k = 0, 1, \dots, n$ we get $\text{mldeg}(X_{W,n}) = 1$ (this is the ML degree of the usual toric variety $\mathcal{X}_n = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^n$). Hence, the question arises whether all integers between 1 and $\deg(\mathcal{X}_n)$ can be realized as $\text{mldeg}(X_{W,n})$. This question has an answer to the affirmative for $\mathbb{P}^m \times \mathbb{P}^n$ for $m = 1, 2, 3$ [CHKO24, Theorem 1.4]. We settle it also for $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^n$.

Theorem 1.8. *For each integer $1 \leq r \leq \deg(\mathcal{X}_n) = (n+1)(n+2)$ there exists $W \in (\mathbb{C}^*)^{4n+4}$ such that $\text{mldeg}(X_{W,n}) = r$.*

Now we give an outline of our paper. Section 2 reviews the relevant background on A -discriminants and the principal A -determinant E_A for toric varieties, and then specializes to the Segre product $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^n$. In Proposition 2.6, we describe exhaustively all possible ways the factors of E_A can vanish for $n = 1$. We will also give a partial description for $n = 2$. Our main goal in Section 3 is to prove Theorem 1.6. This requires a detailed study of the Euler characteristic of the intersection of quadrics in $\mathbb{P}^1 \times \mathbb{P}^1$ associated to f_0, f_1, \dots, f_n as in (1). Along the way we give a formula for the ML degree of the scaled Segre product of $\mathbb{P}^m \times \mathbb{P}^n$ (Corollary 3.4). At the end of the section we provide a counterexample to Conjecture 1.3. Example 3.16 will present two scaling tensors W and W' that lead to two different ML degrees for $n \geq 2$ although exactly the same set of factors of E_A vanish on W and W' . Section 4 gives the complete Euler stratification for $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ (Theorem 1.7). We achieve this by carefully analyzing the intersection patterns of two quadrics in $(\mathbb{C}^*)^2$. Finally, in Section 5, we prove Theorem 1.8 by exhibiting a procedure to construct scaling tensors that produce all possible values for the ML degree of scaled $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^n$.

2 Factors of the Principal A-Determinant

The principal A -determinant E_A plays a crucial role for the Euler stratification of hypersurfaces such as $Y_{W,n}$ defined by (1) (equivalently, for the ML degree stratification of scaled toric varieties such as $X_{W,n}$). In this section, we will define E_A and then study it in the case of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^n$ for $n = 1, 2$. We start with the definition of the A -discriminant; see [GKZ94, Chapter 9].

Definition 2.1. Let $f_w(z) = \sum_{i=1}^N w_i z^{a_i}$ be a polynomial in $\mathbb{C}[z_1, \dots, z_d]$ where $w = (w_1, \dots, w_N) \in (\mathbb{C}^*)^N$, and let $A = [a_1 \ a_2 \ \dots \ a_N] \in \mathbb{Z}^{d \times N}$. Then

$$\nabla_A = \{w \in (\mathbb{C}^*)^N : \exists z \in (\mathbb{C}^*)^d \text{ such that } f_w(z) = \frac{\partial f_w}{\partial z_1} = \dots = \frac{\partial f_w}{\partial z_d} = 0\} \subset \mathbb{C}^N$$

parametrizes hypersurfaces $\{f_w = 0\}$ that have singular points in $(\mathbb{C}^*)^d$. Typically, ∇_A is an irreducible hypersurface and its defining polynomial in $\mathbb{Z}[w_1, \dots, w_N]$ is called the A -discriminant which we denote by $\text{Disc}(A)$. If ∇_A is not a hypersurface we set $\text{Disc}(A) = 1$, in which case we say $\text{Disc}(A)$ is trivial.

Definition 2.2. Let $A \in \mathbb{Z}^{d \times N}$ as above and let $X_A \subset \mathbb{P}^{N-1}$ be the corresponding toric variety. If X_A is smooth, the principal A -determinant is

$$E_A(w) = \prod_{\Gamma \text{ a face of } P} \text{Disc}(\Gamma \cap \mathbb{Z}^d),$$

where $P = \text{conv}(a_1, \dots, a_N)$ and $\text{Disc}(\Gamma \cap \mathbb{Z}^d)$ is the A' -discriminant, where A' is the matrix with columns corresponding to vertices in the face Γ [GKZ94, Chapter 10, Theorem 1.2].

Remark 2.3. In the definition above, every vertex of P contributes a factor w_i to the principal A -determinant. In this paper, we are interested in the vanishing of $E_A(w)$ in $(\mathbb{C}^*)^N$, and since $w \in (\mathbb{C}^*)^N$, we will ignore these factors corresponding to the vertices.

Next we describe the principal A -determinant induced by the hypersurface $Y_{W,n}$. Here, the matrix A is of format $(n+5) \times (4n+4)$, where the columns correspond to the vertices of $P = \Delta_1 \times \Delta_1 \times \Delta_n$ or, equivalently, to the exponents of the terms in f_W in (1).

First we introduce some notation. Note that the coefficient vector W can be viewed as a $2 \times 2 \times (n+1)$ tensor with entries in \mathbb{C}^* . We denote by $W_{\bullet\bullet k}$ the 2×2 matrix

$$W_{\bullet\bullet k} := \begin{pmatrix} w_{00k} & w_{01k} \\ w_{10k} & w_{11k} \end{pmatrix}.$$

Similarly, we define

$$W_{i\bullet(k_1, k_2)} := \begin{pmatrix} w_{i0k_1} & w_{i1k_1} \\ w_{i0k_2} & w_{i1k_2} \end{pmatrix} \quad \text{and} \quad W_{\bullet j(k_1, k_2)} := \begin{pmatrix} w_{0jk_1} & w_{1jk_1} \\ w_{0jk_2} & w_{1jk_2} \end{pmatrix}.$$

The first matrix is a *slice* of W and there are $n+1$ such slices. The next two are 2×2 submatrices of the four *faces* $W_{0\bullet\bullet}$, $W_{1\bullet\bullet}$, $W_{\bullet 0\bullet}$, and $W_{\bullet 1\bullet}$ which are $2 \times (n+1)$ matrices. There are a total of $4 \binom{n+1}{2}$ such matrices. We will call the determinants of all these 2×2 matrices the *2-minors* of W and denote them by

$$F_{\bullet\bullet k} := \det W_{\bullet\bullet k}, \quad F_{i\bullet(k_1, k_2)} := \det W_{i\bullet(k_1, k_2)}, \quad F_{\bullet j(k_1, k_2)} := \det W_{\bullet j(k_1, k_2)}.$$

Furthermore, we will use subtensors of formats $2 \times 2 \times 2$ and $2 \times 2 \times 3$. The first kind is indexed by $0 \leq k_1 < k_2 \leq n$ and we denote it by $W^{k_1 k_2}$. The second kind is indexed by $0 \leq k_1 < k_2 < k_3 \leq n$ and we denote it by $W^{k_1 k_2 k_3}$. There are $\binom{n+1}{2}$ subtensors of the first kind and $\binom{n+1}{3}$ subtensors of the second kind.

Definition 2.4. The $2 \times 2 \times 2$ hyperdeterminant $H_{k_1 k_2}$ of the subtensor $W^{k_1 k_2}$ is

$$H_{k_1 k_2} = [(w_{00k_1} w_{11k_2} - w_{00k_2} w_{11k_1}) - (w_{01k_1} w_{10k_2} - w_{01k_2} w_{10k_1})]^2 - 4 \det(W_{0\bullet(k_1, k_2)}) \det(W_{1\bullet(k_1, k_2)}).$$

The $2 \times 2 \times 3$ hyperdeterminant $H_{k_1 k_2 k_3}$ of the subtensor $W^{k_1 k_2 k_3}$ is equal to the resultant $\text{Res}(q_{k_1}, q_{k_2}, q_{k_3})$ where

$$q_k = w_{00k} x_0 y_0 + w_{01k} x_0 y_1 + w_{10k} x_1 y_0 + w_{11k} x_1 y_1.$$

The $2 \times 2 \times 2$ hyperdeterminants $H_{k_1 k_2}$ are polynomials of degree four in the entries of $W^{k_1 k_2}$ which have 12 terms. The $2 \times 2 \times 3$ hyperdeterminants $H_{k_1 k_2 k_3}$ are polynomials of degree six in the entries of $W^{k_1 k_2 k_3}$ which have 66 terms. See [GKZ94, Chapter 14] and [CHKO24].

Proposition 2.5. [CHKO24, Theorem 1.5] *The principal A -determinant E_A for $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^n$ is the product of all 2-minors of W together with all $2 \times 2 \times 2$ hyperdeterminants $H_{k_1 k_2}$ and all $2 \times 2 \times 3$ hyperdeterminants $H_{k_1 k_2 k_3}$.*

Throughout this paper, we represent subsets of the factors of the principal A -determinant of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^n$ by unions of faces of a tower of n cubes. Each 2-dimensional face of any cube as well as the union of two horizontal edges on the four faces $W_{0\bullet\bullet}, W_{1\bullet\bullet}, W_{\bullet 0\bullet},$ and $W_{\bullet 1\bullet}$ corresponds to a 2-minor of W . The “cube” obtained by a choice of any two slices $W_{\bullet\bullet i}$ and $W_{\bullet\bullet j}$ corresponds to a $2 \times 2 \times 2$ hyperdeterminant factor of E_A . We give examples of this in Figure 1 below.

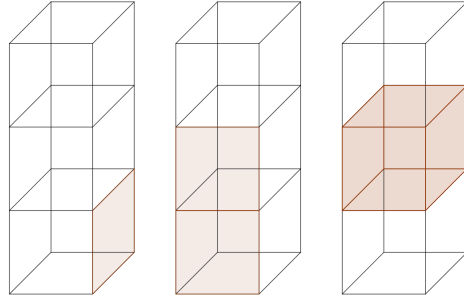


Figure 1: Two different 2×2 determinant factors and one $2 \times 2 \times 2$ hyperdeterminant factor of E_A for $n = 2$

As our main result in this paper we settle Conjecture 1.3. One question this conjecture raises is which subsets of the factors of the principal A -determinant can simultaneously vanish while the rest does not. We answer this question for $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and give some partial results for $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$.

$$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$

In the case of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, we have a $2 \times 2 \times 2$ tensor W with six 2-minors and a single $2 \times 2 \times 2$ hyperdeterminant. Its six 2-minors are $F_{0\bullet\bullet}$, $F_{1\bullet\bullet}$, $F_{\bullet 0\bullet}$, $F_{\bullet 1\bullet}$, $F_{\bullet\bullet 0}$, $F_{\bullet\bullet 1}$, and we denote the unique hyperdeterminant by H . When we say that a subset of the factors of the principal A -determinant E_A vanish, we also mean that the rest of the factors do not vanish.

Proposition 2.6. *The factors of the principal A -determinant E_A for $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ vanish according to the following patterns:*

- a) *Exactly one factor vanishes where there are 7 choices.*
- b) *Exactly two factors vanish where there are $\binom{7}{2}$ choices.*
- c) *Exactly $F_{0\bullet\bullet}$, $F_{\bullet 0\bullet}$, and $F_{\bullet\bullet 0}$ vanish; there are 8 such choices corresponding to 8 "corners" of the $2 \times 2 \times 2$ tensor W .*
- d) *Exactly $F_{0\bullet\bullet}$, $F_{1\bullet\bullet}$, $F_{\bullet 0\bullet}$, $F_{\bullet 1\bullet}$, and H vanish; there are 3 such choices corresponding to two pairs of opposite faces of W .*
- e) *All 7 factors vanish.*

Proof. This can be computed by saturating all possible combinatorial combinations of factors by $\langle \prod_{ijk} w_{ijk} \rangle$ and seeing which other factors are contained in the irreducible components. Alternatively, this is also a corollary of Theorem 1.7 which we will prove in Section 4. We summarize the result in Figure 2. The top row corresponds to cases in a), and the second row corresponds to cases in b). In the third row only the right most configuration is realizable; this corresponds to cases in c). None of the types of configurations in the fourth and sixth rows can be realized. In the fifth row, only the left most configuration is possible; this corresponds to the cases in d). The unique possibility in the bottom row corresponds to the case in e). \square

Remark 2.7. Figure 2 captures all vanishing relations among the factors of the principal A -determinant for $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. A vanishing of a 2-minor is represented by the corresponding face of the cube being shaded in brown, while the vanishing of H is represented by the entire cube being shaded in blue. Each configuration represents multiple such configurations by symmetry. For instance, the first configuration in the top row stands for six such configurations, one for each face of the cube. The configurations are ordered so that every configuration on the same row consists of the same number of factors. We included both realizable and non-realizable configurations. The ones that are highlighted in green correspond to those that are realizable: the vanishing of such a group of factors will not cause any other factors to vanish. If a configuration is obtained by the addition of a single factor from another realizable configuration, a black arrow is included between the two configurations. A red arrow indicates an implication of vanishing of factors: if the vanishing of a set of factors implies the vanishing of further factors, we placed a red arrow between the corresponding configurations.

Remark 2.8. By Proposition 1.2, the complement of the real hypersurface arrangement in $(\mathbb{R}^*)^8$ defined by $E_A(W) = 0$ consists of $W \in (\mathbb{R}^*)^8$ where $\chi(Y_{W,1}) = 6$. Each region in the complement is determined by which side of the hypersurfaces defined by the six minors and the hyperdeterminant it lies on, positive or negative. Hence, there are $2^7 = 128$ possible sign patterns in this case. A simple computation using `HypersurfaceRegions.jl` shows that 68 of these sign patterns are realizable. Out of these, $64 = \frac{128}{2}$ are all possible sign patterns corresponding to regions in H^+ . The remaining four are

$+++++-$
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where signs correspond to $F_{0\bullet\bullet}, F_{1\bullet\bullet}, F_{\bullet 0\bullet}, F_{\bullet 1\bullet}, F_{\bullet\bullet 0}, F_{\bullet\bullet 1}$, and H , in that order.

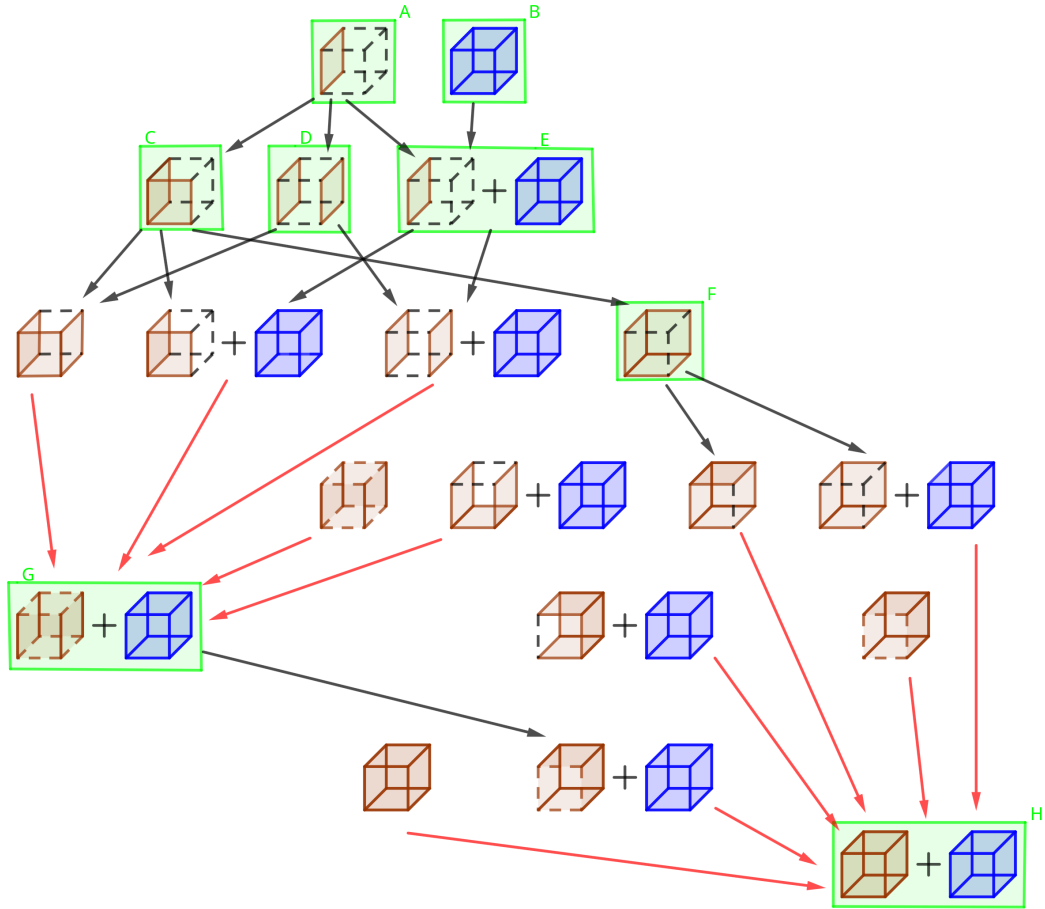


Figure 2: Vanishing relations among factors of E_A for the case $n = 1$

$$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$$

A complete classification of the vanishing patterns of the factors of E_A in the case of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ is out of our reach. Here we briefly report the cases with two or three vanishing factors which give rise to further factors vanishing. Recall that now W is a $2 \times 2 \times 3$ tensor. For ease of notation, we first introduce the following.

Definition 2.9. 1. Let $F = \{F_1, F_2\}$ be a set of two minors not on the same face of W . We say F is a *hook* if the minors F_1 and F_2 share exactly two variables.

2. Let $F = \{F_1, F_2\}$ be as above. We say F is a *mirror* if

- there is a $2 \times 2 \times 2$ subtensor of W containing F and
- the variables in F_1 and F_2 are disjoint.

3. Let $F = \{F_1, F_2, F_3\}$ be a set of three distinct minors of W . Then F is a *square cup* if

- there is a $2 \times 2 \times 2$ subtensor of W containing F and
- the variables in F_1 and F_2 are disjoint.

4. Let $F = \{F_1, F_2, F_3, F_4\}$ be a set of distinct minors of W . We say F is a *cubic frame* if

- there is a $2 \times 2 \times 2$ subtensor of W containing F ,
- the variables in F_1 and F_2 are disjoint and
- the variables in F_3 and F_4 are disjoint.

In the list below we describe (almost) all configurations with two or three vanishing factors which imply further factors vanishing. For this subsection only, we denote the unique $2 \times 2 \times 3$ hyperdeterminant by H . We studied all possible combinatorial cases as illustrated in the following example:

Example 2.10. Assume the vanishing of a mirror with minors contained in two 2×3 faces of W , say $F_{0\bullet(0,1)} = F_{1\bullet(0,1)} = 0$ as well as $H = 0$. The saturation of this ideal by $\langle \prod_{ijk} w_{ijk} \rangle$ has four minimal primes. The first one contains $F_{0\bullet(0,2)}$ and $F_{0\bullet(1,2)}$, the second one contains $F_{1\bullet(0,2)}$ and $F_{1\bullet(1,2)}$, the third one contains $F_{\bullet 0(0,1)}$, $F_{\bullet 1(0,1)}$ and H_{01} and the fourth one contains $F_{\bullet\bullet 0}$, $F_{\bullet\bullet 1}$ and H_{01} . Thus, either one of the sides $W_{0\bullet\bullet}$ or $W_{1\bullet\bullet}$ has all its minors vanish, or H_{01} as well as one of the cubic frames containing the mirror $F_{0\bullet(0,1)}$, $F_{1\bullet(0,1)}$ vanishes. This is precisely case 5 in the list below.

1. Two minors within the same face of W vanish \implies all three minors within that face and H vanish.
2. The minors in a square cup vanish \implies the minors of the cubic frame containing the square cup in the same $2 \times 2 \times 2$ subtensor, its hyperdeterminant, as well as H vanish.
3. The minors in a hook and the $2 \times 2 \times 2$ -hyperdeterminant of the corresponding subtensor vanish \implies the minors of the cubic frame containing the hook and H vanish as well.

4. The minors in a mirror and the $2 \times 2 \times 2$ -hyperdeterminant of the corresponding subtensor vanish \implies the minors in one of the cubic frames containing the mirror and H vanish as well.
5. A mirror with minors contained in two 2×3 faces $\{S_1, S_2\}$ of W and H vanish \implies
 - all minors within S_1 or S_2 vanish or
 - the minors in one of the cubic frames containing the mirror and the hyperdeterminant of the corresponding $2 \times 2 \times 2$ subtensor vanish.
6. A hook with minors contained in two adjacent 2×3 faces $\{S_1, S_2\}$ of W and H vanish \implies
 - all minors within S_1 or S_2 vanish or
 - the minors in the cubic frame containing the hook and the hyperdeterminant of the corresponding the $2 \times 2 \times 2$ subtensor vanish.
7. A $2 \times 2 \times 2$ -hyperdeterminant G , a minor U coming from a slice of W contained in the corresponding subtensor of G and H vanish \implies
 - the $2 \times 2 \times 2$ -hyperdeterminant of the other subtensor containing U vanish or
 - the minors in one of the cubic frames containing U vanish.
8. A $2 \times 2 \times 2$ -hyperdeterminant G , a minor U coming from a 2×3 face S of W and contained in the corresponding subtensor of G and H vanish \implies
 - all minors within S vanish or
 - the minors in one of the cubic frames containing U vanish.
9. Two $2 \times 2 \times 2$ -hyperdeterminants and the minor fully contained in both corresponding subtensors vanish $\implies H$ vanishes as well.

This list is complete except for two cases, for which the computations were too expensive. We do not know if there are any implications if

- all three $2 \times 2 \times 2$ -hyperdeterminants vanish or
- two of the $2 \times 2 \times 2$ -hyperdeterminants and H vanish.

Additionally, the following result will be useful for proving Theorem 1.8.

Lemma 2.11. *A set of minors vanishing will cause H to vanish if and only if that set contains either a square cup or two out of three minors of a face of W .*

Proof. This can be seen by exhaustive computation. □

3 ML degree of three-way independence models

In this section we start with observations regarding the ML degree of general scaled toric varieties. Let $f(x, z)$ be a Laurent polynomial of the form

$$f(x, z) = f_0(x) + \sum_{k=1}^n z_k f_k(x),$$

where $f_0, \dots, f_n \in \mathbb{C}[x_1^{\pm 1}, \dots, x_m^{\pm 1}]$ are Laurent polynomials in m variables. Let $A \subseteq \mathbb{Z}^{n+m}$ denote the set of exponent vectors of the terms in $f(x, z)$, and let $W = (w_a \in \mathbb{C}^* : a \in A)$ be the coefficients of these terms. The scaled toric variety $X_{A,W} \subseteq \mathbb{P}^{|A|-1}$ is given by the monomial parametrization $(\mathbb{C}^*)^{n+m} \mapsto \mathbb{P}^{|A|-1}$

$$(t_1, \dots, t_{n+m}) \mapsto [w_a t^a : a \in A] \in \mathbb{P}^{|A|-1}.$$

As in the case of Proposition 1.1, it is known from [Huh13, Theorem 1] and [TW24, Section 6.4] that the ML degree of $X_{A,W}$ is equal to

$$\text{mldeg}(X_{A,W}) = -(-1)^{n+m} \chi(V(f) \cap (\mathbb{C}^*)^{n+m}),$$

where $V(f)$ is the affine hypersurface defined by the vanishing of $f(x, z)$. Our starting point is the following result.

Theorem 3.1. [FMH24, Theorem 2.2] *Let $f(x, z) = f_0(x) + \sum_{k=1}^n z_k f_k(x)$ be a Laurent polynomial where $f_0, \dots, f_n \in \mathbb{C}[x_1^{\pm 1}, \dots, x_m^{\pm 1}]$. Then*

$$\chi(V(f) \cap (\mathbb{C}^*)^{m+n}) = (-1)^n \chi(V(f_0 f_1 \cdots f_n) \cap (\mathbb{C}^*)^m).$$

Throughout the rest of the paper, for any positive integer k , we will use the notation $[\bar{k}] := \{0, 1, \dots, k\}$. Now, for $\emptyset \neq I \subseteq [\bar{n}]$ let us denote by C_I the intersection in the torus

$$C_I = V(f_k : k \in I) \subseteq (\mathbb{C}^*)^m.$$

Via elementary exclusion-inclusion, one can go from the union $V(f_0 \dots f_n)$ to intersections and obtain the following statement.

Corollary 3.2. *The Euler characteristic of $V(f) \cap (\mathbb{C}^*)^{m+n}$ is equal to*

$$\chi(V(f) \cap (\mathbb{C}^*)^{m+n}) = (-1)^{n+1} \sum_{\emptyset \neq I \subseteq [\bar{n}]} (-1)^{|I|} \chi(C_I),$$

and therefore the ML degree of $X_{A,W}$ is equal to

$$\text{mldeg}(X_{A,W}) = (-1)^m \sum_{\emptyset \neq I \subseteq [\bar{n}]} (-1)^{|I|} \chi(C_I).$$

Now consider $f(x, z)$ such that the set of its exponent vectors is the vertex set of a product of $r+1$ simplices $\Delta_{d_1} \times \dots \times \Delta_{d_r} \times \Delta_n$. The last n -dimensional simplex corresponds to the coordinates z_1, \dots, z_n in $f(x, z)$. Instead of considering the intersections C_I in the torus $(\mathbb{C}^*)^{d_1} \times \dots \times (\mathbb{C}^*)^{d_r}$, we would like to consider them in the product of projective spaces $\mathbb{P}^{d_1} \times \dots \times \mathbb{P}^{d_r}$. We denote the homogenous coordinates of the factor \mathbb{P}^{d_i} by $x_0^{(i)}, \dots, x_{d_i}^{(i)}$. In these coordinates, the homogenized versions of $f_k(x)$ will be denoted by

$$q_k(x_0^{(1)}, \dots, x_{d_1}^{(1)}, \dots, x_0^{(r)}, \dots, x_{d_r}^{(r)}).$$

Note that with these coordinates we can organize the coefficients of $f(x, z)$ into a tensor $W = (w_{i_1 \dots i_r k})$ of format $(d_1 + 1) \times \dots \times (d_r + 1) \times (n + 1)$ where $w_{i_1 \dots i_r k}$ is the coefficient of $x_{i_1}^{(1)} \dots x_{i_r}^{(r)} z_k$. Let V_I denote now the intersection in $\mathbb{P}^{d_1} \times \dots \times \mathbb{P}^{d_r}$ defined by

$$V_I = V(q_k : k \in I).$$

For a subset $\mathcal{J} = J_1 \times \dots \times J_r \subseteq [\bar{d}_1] \times \dots \times [\bar{d}_r]$ let us introduce the notation

$$X_{\mathcal{J}} = \left\{ x_j^{(i)} = 0 : j \in J_i \right\} \subseteq \mathbb{P}^{d_1} \times \dots \times \mathbb{P}^{d_r}.$$

With this we get a formula for the ML degree of the scaled toric variety associated to a product of simplices as in Corollary 3.2.

Theorem 3.3. *Let A be the vertex set of the product of simplices $\Delta_{d_1} \times \dots \times \Delta_{d_r} \times \Delta_n$ and W be a scaling considered as a tensor of format $(d_1 + 1, \dots, d_r + 1, n + 1)$. Then*

$$\text{mldeg}(X_{A,W}) = (-1)^{\sum_{i=1}^r d_i} \sum_{\emptyset \neq I \subseteq [\bar{n}]} (-1)^{|I|} \left[\sum_{\mathcal{J}: J_i \subseteq [\bar{d}_i]} (-1)^{|\mathcal{J}|} \chi(V_I \cap X_{\mathcal{J}}) \right],$$

where $|\mathcal{J}| := |J_1| + \dots + |J_r|$.

Proof. We start with the identity $C_I = V_I \cap \{x_{j_i}^{(i)} \neq 0 : i = 1, \dots, r, j_i = 0, \dots, d_i\}$. The second term in the intersection is equal to

$$\mathbb{P}^{d_1} \times \dots \times \mathbb{P}^{d_r} \setminus \bigcup_{i=1}^r \bigcup_{j=0}^{d_i} \{x_j^{(i)} = 0\}.$$

Inclusion-exclusion combined with Corollary 3.2 gives the result. \square

We note that with Theorem 3.3 we managed to express the ML degree in terms $\chi(V_I \cap X_{\mathcal{J}})$ and these only depend on the subtensors of W that have indices belonging to the set $([\bar{d}_1] \setminus J_1) \times \dots \times ([\bar{d}_r] \setminus J_r) \times I$. This has a nice consequence for the product of two simplices; see also [CHKO24].

Corollary 3.4. *The ML degree of a product of two simplices $\Delta_m \times \Delta_n$ with the scaling W considered as an $(m + 1) \times (n + 1)$ matrix is equal to*

$$\text{mldeg}(X_{\Delta_m \times \Delta_n, W}) = \sum_{\substack{\emptyset \neq J \subseteq [\bar{m}] \\ \emptyset \neq I \subseteq [\bar{n}]}} (-1)^{|I|+|J|} \text{rank}(W_{IJ}),$$

where W_{IJ} is the submatrix whose rows and columns are indexed by I and J , respectively.

Proof. Theorem 3.3 gives us

$$\text{mldeg}(X_{\Delta_m \times \Delta_n, W}) = (-1)^m \sum_{\emptyset \neq I \subseteq [\bar{n}]} (-1)^{|I|} \left[\sum_{J \subseteq [\bar{m}]} (-1)^{|J|} \chi(V_I \cap X_J) \right].$$

The intersection $V_I \cap X_J$ is a linear subspace in $\mathbb{P}^{m-|J|}$ defined by the vanishing of linear forms whose coefficients are the entries of $W_{IJ'}$, where J' is the complement of J in $[\bar{m}]$. The Euler characteristic of this linear subspace is precisely $m - |J| + 1 - \text{rank}(W_{IJ'})$. Now using the fact that $\sum_{k=0}^m (-1)^k \binom{m+1}{k} \cdot (m+1-k)$ is zero and swapping J and J' we arrive at the formula. \square

3.1 Scaled $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^n$

Now we specialize back to our main focus of the paper, namely three-way independence models based on $\Delta_1 \times \Delta_1 \times \Delta_n$. Our goal is to prove Theorem 1.6, and in order to do so, we are going to understand how the terms in Theorem 3.3 behave with respect to the pattern of vanishing of the factors of the principal A -determinant E_A . Theorem 1.6 is the combination of Corollary 3.10 and Corollary 3.14 below. Along the way, we will give explicit expressions for many terms in the corresponding formula of Theorem 3.3.

We go back to the notation from earlier sections where $x_i^{(1)} = x_i$ and $x_i^{(2)} = y_i$, and let

$$q_k(x_0, x_1, y_0, y_1) = w_{00k}x_0y_0 + w_{01k}x_0y_1 + w_{10k}x_1y_0 + w_{11k}x_1y_1.$$

The terms in Theorem 3.3 for our case can be subdivided into two types:

$$(i) \chi(V_I \cap X_{\mathcal{J}}) \text{ with } 1 \leq |\mathcal{J}| \leq 2 \quad \text{and} \quad (ii) \chi(V_I).$$

Lemma 3.5. *The Euler characteristic $\chi(V_I \cap X_{\mathcal{J}})$ with $1 \leq |\mathcal{J}| \leq 2$ depends only on which 2-minors of W vanish.*

Proof. If $|\mathcal{J}| = 2$, then $X_{\mathcal{J}}$ is just a point in $\mathbb{P}^1 \times \mathbb{P}^1$. Without loss of generality, it is given by $x_0 = y_0 = 0$. Substituting this into q_k for $k \in I$, we see that $V_I \cap X_{\mathcal{J}} = \emptyset$ since all coefficients in W are in \mathbb{C}^* . This shows that the Euler characteristic is 0.

If $|\mathcal{J}| = 1$, then $X_{\mathcal{J}} = \text{pt} \times \mathbb{P}^1$ or $\mathbb{P}^1 \times \text{pt}$. Therefore, we are looking at the intersection of $|I|$ linear hyperplanes in \mathbb{P}^1 . For example, for $\mathcal{J} = \{1-i\} \times \emptyset$, as in the proof of Corollary 3.4 we get

$$\chi(V_I \cap \{x_{1-i} = 0\}) = 2 - \text{rank} \begin{pmatrix} w_{i0k_1} & w_{i1k_1} \\ \vdots & \vdots \\ w_{i0k_{|I|}} & w_{i1k_{|I|}} \end{pmatrix}.$$

The matrix above is a $|I| \times 2$ matrix lying on the face $W_{i\bullet\bullet}$. Similarly for $V_I \cap \{y_{1-j} = 0\}$ one gets a $|I| \times 2$ matrix on the face $W_{\bullet j\bullet}$. Since we assume that $W \in (\mathbb{C}^*)^{4n+4}$, the vanishing pattern of the 2-minors completely governs the ranks of the above matrices. \square

The remainder of this section is devoted to dealing with $\chi(V_I)$. With $I = \{k_1, \dots, k_{|I|}\}$, the system $q_{k_1} = \dots = q_{k_{|I|}} = 0$ can be written as

$$0 = T_I(y) \cdot x := \begin{pmatrix} w_{00k_1}y_0 + w_{01k_1}y_1 & w_{10k_1}y_0 + w_{11k_1}y_1 \\ \vdots & \vdots \\ w_{00k_{|I|}}y_0 + w_{01k_{|I|}}y_1 & w_{10k_{|I|}}y_0 + w_{11k_{|I|}}y_1 \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}.$$

Note that for a fixed $y \in \mathbb{P}^1$, this system admits a solution $x \in \mathbb{P}^1$ if and only if $\text{rank } T_I(y) < 2$.

The cases with $1 \leq |I| \leq 2$

We start with the case $|I| = 1$.

Lemma 3.6. *For $W \in (\mathbb{C}^*)^{4n+4}$, the Euler characteristic $\chi(V_{\{k\}})$ depends only on whether $F_{\bullet\bullet k}$ vanishes. More precisely, $\chi(V_{\{k\}}) = 4 - \text{rank } W_{\bullet\bullet k}$.*

Proof. In this situation the rank of $T_k(y)$ is always at most 1, but there might exist a point y where the rank drops to 0. This happens if and only if $w_{00k}y_0 + w_{01k}y_1 = \lambda(w_{10k}y_0 + w_{11k}y_1)$ for $\lambda \in \mathbb{C}^*$. In turn, this happens if and only if $\det W_{\bullet\bullet k} = F_{\bullet\bullet k} = 0$. When $\text{rank } W_{\bullet\bullet k} = 2$, $V_{\{k\}}$ is of the form $\mathbb{P}^1 \times \text{pt}$, and when $\text{rank } W_{\bullet\bullet k} = 1$, $V_{\{k\}}$ is of the form $((\mathbb{P}^1 \setminus \text{pt}) \times \text{pt}) \sqcup (\text{pt} \times \mathbb{P}^1)$. In both cases, $\chi(V_{\{k\}}) = 4 - \text{rank } W_{\bullet\bullet k}$. \square

For the case $|I| = 2$, we let $I = \{i, j\}$. We can still express the contribution $\chi(V_I)$ explicitly, but it is a little bit more involved. For a solution in $\mathbb{P}^1 \times \mathbb{P}^1$ to $T_{ij}(y) \cdot x = 0$ to exist, $\det T_{ij}(y)$ must vanish.

Lemma 3.7. *The determinant of $T_{ij}(y)$ is*

$$\det T_{ij}(y) = F_{\bullet 0(i,j)} y_0^2 + (w_{00i}w_{11j} + w_{01i}w_{10j} - w_{10i}w_{01j} - w_{11i}w_{00j}) y_0 y_1 + F_{\bullet 1(i,j)} y_1^2,$$

and the discriminant of this quadric is exactly the hyperdeterminant H_{ij} .

The set $\{y \in \mathbb{P}^1 : \det T_{ij}(y) = 0\}$ can consist of 1 or 2 points or be the whole \mathbb{P}^1 . However, to understand the geometry of V_I , it is not enough to just know the points where $\det T_{ij}(y) = 0$, we also need to know the rank of $T_{ij}(y)$ at those points. The matrix $T_{ij}(y)$ can have rank 0 at some point y only if it factorizes as

$$T_{ij}(y) = (a_0 y_0 + a_1 y_1) \cdot M, \tag{4}$$

with $a_i \in \mathbb{C}$ and M a constant 2×2 matrix. We note that in this case the corresponding hyperdeterminant H_{ij} vanishes.

If $T_{ij}(y)$ does not factorize as above, then there are a few possible scenarios. If H_{ij} does not vanish, then $\det T_{ij}(y) = 0$ at exactly two points. Otherwise, when H_{ij} does vanish, we can have either one point at which $\text{rank } T_{ij}(y) = 1$, since H_{ij} is the discriminant of the quadratic equation $\det T_{ij}(y) = 0$, or $\text{rank } T_{ij}(y) = 1$ for any point y , in which case all the coefficients of the equation $\det T_{ij}(y) = 0$ are identically zero.

If the rank of $T_{ij}(y)$ is 1 at some y , we find exactly one $x \in \mathbb{P}^1$ to satisfy $T_{ij}(y) \cdot x = 0$. If the rank of $T_{ij}(y)$ is 0 at some y , all $x \in \mathbb{P}^1$ satisfy $T_{ij}(y) \cdot x = 0$. We summarize this discussion as follows.

Lemma 3.8. *The matrix $T_{ij}(y)$ falls into one of the following five types.*

I: $T_{ij}(y)$ has rank 1 at two points and never rank 0.

This happens if the hyperdeterminant H_{ij} does not vanish. Then $T_{ij}(y) \cdot x = 0$ has two solutions $\text{pt} \times \text{pt} \sqcup \text{pt} \times \text{pt}$ and $\chi(V_I) = 2$.

II: $T_{ij}(y)$ **has rank 0 at one point only and rank < 2 nowhere else.**

This happens if $T_{ij}(y) = (a_0y_0 + a_1y_1)M$ with $\text{rank } M = 2$. Then $T_{ij}(y) \cdot x = 0$ has as solution $\mathbb{P}^1 \times \text{pt}$ and $\chi(V_I) = 2$.

III: $T_{ij}(y)$ **has rank 0 at one point only and rank 1 everywhere else.**

This happens if $T_{ij}(y) = (a_0y_0 + a_1y_1)M$ with $\text{rank } M = 1$. Then the set of solutions to $T_{ij}(y) \cdot x = 0$ has the form $(\mathbb{P}^1 \times \text{pt}) \sqcup \text{pt} \times (\mathbb{P}^1 \setminus \text{pt})$ and $\chi(V_I) = 3$.

IV: $T_{ij}(y)$ **has always rank 1 and never rank 0.**

This happens if $\det T_{ij}(y)$ is identically zero. Then the solution set to $T_{ij}(y) \cdot x = 0$ is isomorphic to \mathbb{P}^1 and $\chi(V_I) = 2$.

V: $T_{ij}(y)$ **has rank 1 at one point only.**

This happens if $\det T_{ij}(y)$ is not identically zero but the hyperdeterminant H_{ij} vanishes. Then $T_{ij}(y) \cdot x = 0$ has one solution $\text{pt} \times \text{pt}$ and $\chi(V_I) = 1$.

We want to understand what kind of coefficient tensor W corresponds to each of the above types. For a 3-way tensor $W = (w_{ijk}) \in \mathbb{C}^p \times \mathbb{C}^q \times \mathbb{C}^r$ and $s = 1, 2, 3$, we let $W^{(s)}$ be the flattening matrix along mode s . For instance, $W^{(1)}$ is the $p \times (qr)$ matrix where the entry in row i and column indexed by (j, k) is w_{ijk} . In particular for the $2 \times 2 \times 2$ tensor $W_{\bullet\bullet(i,j)}$ we denote the corresponding flattenings by $W_{ij}^{(s)}$.

Lemma 3.9. *Let $W \in (\mathbb{C}^*)^{4n+4}$. The type of the matrix $T_{ij}(y)$ depends only on the vanishing of the 2-minors of $W_{\bullet\bullet(i,j)}$ and its $2 \times 2 \times 2$ hyperdeterminant H_{ij} . The matrix $T_{ij}(y)$ is of type*

I: if $H_{ij} \neq 0$,

II: if $H_{ij} = 0$ and $F_{\bullet\bullet i} = F_{\bullet\bullet j} = F_{0\bullet(i,j)} = F_{1\bullet(i,j)} = 0$ and $F_{\bullet 0(i,j)} \neq 0, F_{\bullet 1(i,j)} \neq 0$,

III: if $H_{ij} = 0$ and all 2-minors of $W_{\bullet\bullet(i,j)}$ vanish,

IV: if $H_{ij} = 0$ and

either $F_{\bullet\bullet i} = F_{\bullet\bullet j} = F_{\bullet 0(i,j)} = F_{\bullet 1(i,j)} = 0$ and $F_{0\bullet(i,j)} \neq 0, F_{1\bullet(i,j)} \neq 0$

or $F_{0\bullet(i,j)} = F_{1\bullet(i,j)} = F_{\bullet 0(i,j)} = F_{\bullet 1(i,j)} = 0$ and $F_{\bullet\bullet i} \neq 0, F_{\bullet\bullet j} \neq 0$,

V: if $H_{ij} = 0$ and only one of the 2-minors of $W_{\bullet\bullet(i,j)}$ vanishes.

Proof. As already mentioned, if the hyperdeterminant $H_{ij} \neq 0$ then $T_{ij}(y)$ is of type I. Assume from now on that $H_{ij} = 0$.

The matrix $T_{ij}(y)$ factors if and only if the tuple $(w_{00i} \ w_{00j} \ w_{10i} \ w_{10j})$ is proportional to the tuple $(w_{01i} \ w_{01j} \ w_{11i} \ w_{11j})$ or in other words the matrix

$$W_{ij}^{(2)} = \begin{pmatrix} w_{00i} & w_{00j} & w_{10i} & w_{10j} \\ w_{01i} & w_{01j} & w_{11i} & w_{11j} \end{pmatrix}$$

has rank 1. This happens if and only if $F_{\bullet\bullet i} = F_{\bullet\bullet j} = F_{0\bullet(i,j)} = F_{1\bullet(i,j)} = 0$. Hence we have a clear way to distinguish types II and III from the types IV and V.

To decide whether $T_{ij}(y)$ falls into type II or III, one needs to compute the rank of the factor M from (4). We see that $\text{rank } M = 1$ if and only if $F_{\bullet 0(i,j)} = F_{\bullet 1(i,j)} = 0$, since

$$\text{rank } M = \text{rank} \begin{pmatrix} w_{00i} & w_{10i} \\ w_{00j} & w_{10j} \end{pmatrix} = \text{rank} \begin{pmatrix} w_{01i} & w_{11i} \\ w_{01j} & w_{11j} \end{pmatrix}.$$

Now assume that the factorization (4) does not happen. The rank of $T_{ij}(y)$ can be 1 for any y (in which case we obtain type IV) if either the rows or columns of $T_{ij}(y)$ are proportional. Thus either

$$\begin{aligned} \text{rank } W_{ij}^{(1)} &= \text{rank} \begin{pmatrix} w_{00i} & w_{01i} & w_{00j} & w_{01j} \\ w_{10i} & w_{11i} & w_{10j} & w_{11j} \end{pmatrix} = 1 \quad \text{or} \\ \text{rank } W_{ij}^{(3)} &= \text{rank} \begin{pmatrix} w_{00i} & w_{01i} & w_{10i} & w_{11i} \\ w_{00j} & w_{01j} & w_{10j} & w_{11j} \end{pmatrix} = 1. \end{aligned}$$

The first matrix has rank 1 if and only if $F_{\bullet\bullet i} = F_{\bullet\bullet j} = F_{\bullet 0(i,j)} = F_{\bullet 1(i,j)} = 0$. The second matrix has rank 1 if and only if $F_{0\bullet(i,j)} = F_{1\bullet(i,j)} = F_{\bullet 0(i,j)} = F_{\bullet 1(i,j)} = 0$. Otherwise, the matrix $T_{ij}(y)$ must be of type V. \square

Since for $n = 1$ the size of I is at most 2, we have covered everything needed to compute the Euler characteristic $\chi(Y_{W,1})$

Corollary 3.10 (Theorem 1.6, part (1)). *If exactly the same factors of the principal A -determinant E_A vanish on W and W' , then $\chi(Y_{W,1}) = \chi(Y_{W',1})$.*

The cases with $|I| \geq 3$

Now we can deal with the cases $|I| \geq 3$. If $|I| = 3$ and the corresponding $2 \times 2 \times 3$ hyperdeterminant does not vanish, then the resultant $\text{Res}(q_k : k \in I)$ is nonzero, and hence $V_I = \emptyset$. More generally, we can say the following.

Lemma 3.11. *For an index set $I \subseteq [\bar{n}]$ with $|I| \geq 3$, if one of the $2 \times 2 \times 3$ hyperdeterminants H_{ijk} with $i, j, k \in I$ does not vanish, then $V_I = \emptyset$ and $\chi(V_I) = 0$. For $|I|=3$, $V_I \neq \emptyset$ if and only if $H_{ijk} = 0$.*

However, for $|I| \geq 4$, $V_I \neq \emptyset$ is not equivalent to the vanishing of all $2 \times 2 \times 3$ hyperdeterminants H_{ijk} with $i, j, k \in I$. We do not have anything further to report in this case, so we will treat only the case of $|I| = 3$ in the remainder.

For $I = \{k_1, k_2, k_3\}$ and $H_{k_1 k_2 k_3} = 0$, in order to understand the geometry of V_I , we exploit the fact that

$$V_{\{k_1, k_2, k_3\}} = V_{\{k_1, k_2\}} \cap V_{\{k_1, k_3\}} \cap V_{\{k_2, k_3\}},$$

and our results about $V_{\{i,j\}}$. A special role is played by the flattening

$$W_I^{(3)} = \begin{pmatrix} w_{00k_1} & w_{01k_1} & w_{10k_1} & w_{11k_1} \\ w_{00k_2} & w_{01k_2} & w_{10k_2} & w_{11k_2} \\ w_{00k_3} & w_{01k_3} & w_{10k_3} & w_{11k_3} \end{pmatrix}$$

of the $2 \times 2 \times 3$ subtensor $W_{\bullet\bullet(k_1, k_2, k_3)}$.

Remark 3.12. Note that the rank of the above matrix $W_I^{(3)}$ is exactly what defines the singular locus of the $2 \times 2 \times 3$ hyperdeterminant $H_{k_1 k_2 k_3}$ (see Proposition 5.4.(c) in [WZ96]). Namely, the singular locus is exactly the set of W for which $\text{rank } W_I^{(3)} < 3$. This locus cannot be described by the vanishing of certain factors of the principal A -determinant (see Example 3.16 later).

For the $2 \times 2 \times 2$ -hyperdeterminants H_{ij} the singular locus is given by the set of W where the flattenings $W_{ij}^{(2)}$ have rank ≤ 1 (see the end of Section 2 in loc.cit.) As we already noted, if we consider W only with entries in \mathbb{C}^* , then this is governed by the vanishing of the 2-minors of the corresponding $2 \times 2 \times 2$ -tensor.

Proposition 3.13. *Let $W \in (\mathbb{C}^*)^{4n+4}$ and consider $I = \{k_1, k_2, k_3\}$. Then, the Euler characteristic $\chi(V_I)$ depends only on the vanishing of the factors of E_A if $H_{k_1 k_2 k_3}$ does not vanish at W or if one of $H_{ij} = 0$ with $i, j \in I$. Otherwise, $\chi(V_I)$ depends on the factors of E_A that vanish at W , as well as whether $\text{rank } W_I^{(3)} = 3$.*

The above proposition and Lemma 3.11 together with the results for $|I| \leq 2$ combine to give the remaining parts of Theorem 1.6.

Corollary 3.14 (Theorem 1.6, parts (2) and (3)). *If exactly the same factors of E_A vanish on W and W' , then $\chi(Y_{W,n}) = \chi(Y_{W',n})$ as long as*

- $n \geq 2$ and at least one of the $2 \times 2 \times 3$ hyperdeterminants $H_{ijk} \neq 0$ with $\{i, j, k\} \subseteq [\bar{n}]$
- $n = 2$ and at least one of H_{01}, H_{02}, H_{12} vanishes.

It is instructive to write down the implications for $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$.

Corollary 3.15. *Let $n = 2$ and $W \in (\mathbb{C}^*)^{12}$. Then $\chi(Y_{W,2})$ depends only on the factors of E_A that vanish at W and on whether the flattening $W^{(3)}$ has full rank.*

Proof of Proposition 3.13. Without loss of generality we assume that $I = \{0, 1, 2\}$. If $H_{012} \neq 0$, then $\chi(V_I) = 0$. In the remainder we assume that $H_{012} = 0$ and V_I is not empty. We focus on the matrices $T_{ij}(y)$ for $i, j \in I$. We use Lemma 3.8 extensively.

If at least one matrix $T_{ij}(y)$ is of type V, then it follows that V_{ij} , and thus V_I , consists of exactly one point with $\chi(V_I) = 1$. From now on assume that no $T_{ij}(y)$ is of type V.

Now suppose that one of the matrices, say $T_{01}(y)$, is of type II. Therefore there exists exactly one $y \in \mathbb{P}^1$ such that $\text{rank } T_{01}(y) < 2$. In order for the system $T_I(y) \cdot x = 0$ to have a solution (as we assume it does), the other two matrices must also drop rank at y . If the matrices $T_{02}(y)$ and $T_{12}(y)$ are both of types II or III, then $T_I(y) \cdot x = 0$ has solution of the form $\mathbb{P}^1 \times \text{pt}$ and $\chi(V_I) = 2$. If at least one of $T_{02}(y)$ and $T_{12}(y)$ is of type I or IV, then $T_I(y) \cdot x = 0$ has a solution of the form $\text{pt} \times \text{pt}$ and $\chi(V_I) = 1$. Now we assume that no $T_{ij}(y)$ is of type II.

If all matrices $T_{ij}(y)$ are of type III, then their factorizations must coincide and thus $\chi(V_I) = 3$. If at least one is of type IV and the rest is of type III, then there is at least one $T_{ij}(y)$ that has rank 1 at each point $y \in \mathbb{P}^1$. Since the other matrices also have either rank 1 or 0 at each y , $T_I(y)$ has rank 1 for each y , therefore, for each y there is exactly one solution of $T_I(y) \cdot x = 0$ and $\chi(V_I) = 2$. From now on assume that at least one $T_{ij}(y)$ is of type I.

Note that it is not possible to have exactly two matrices $T_{ij}(y)$ of type III since this would force the third one to be of this type also. Suppose we have only one matrix of type I, say $T_{01}(y)$. Then the types of our three matrices are either I,III,IV or I, IV,IV. Let y^1, y^2 be the two points giving $\text{rank } T_{01}(y^i) = 1$. In both possible cases, we have $\text{rank } T_I(y^i) = 1$, and thus V_I is of the form $\text{pt} \times \text{pt} \sqcup \text{pt} \times \text{pt}$ and $\chi(V_I) = 2$.

Now suppose that $T_{01}(y)$ and $T_{02}(y)$ are of type I. Then the quadratic equations $\det T_{01}(y) = 0$ and $\det T_{02}(y) = 0$ have exactly two solutions each and we can write

$$\begin{aligned}\det T_{01}(y) &= (a_{01}y_0 + b_{01}y_1)(c_{01}y_0 + d_{01}y_1), \\ \det T_{02}(y) &= (a_{02}y_0 + b_{02}y_1)(c_{02}y_0 + d_{02}y_1)\end{aligned}$$

with $(a_{0i}, b_{0i}) \neq (c_{0i}, d_{0i})$. Since we assume that V_I is nonempty, one or two factors between $\det T_{01}(y)$ and $\det T_{02}(y)$ must coincide.

If $T_{12}(y)$ is of type IV with proportional rows or of type III, which also has proportional rows, then there exists $\mu \in \mathbb{C}^*$ such that $\det T_{01}(y) = \mu \det T_{02}(y)$. Thus, both factors coincide up to multiplicity and $\chi(V_I) = 2$.

If $T_{12}(y)$ is of type IV with columns proportional by a number μ , then we have

$$\begin{aligned}\det T_{01}(y) &= (w_{001}y_0 + w_{011}y_1)((\mu w_{000} - w_{100})y_0 + (\mu w_{010} - w_{110})y_1) \\ \det T_{02}(y) &= (w_{002}y_0 + w_{012}y_1)((\mu w_{000} - w_{100})y_0 + (\mu w_{010} - w_{110})y_1).\end{aligned}$$

One factor is the same, but the other is not since we assume that the rows of $T_{12}(y)$ are not proportional. Hence, in this case, V_I is a single point and $\chi(V_I) = 1$. Note here that a matrix of type IV has either its rows or its columns proportional, but never both. Which one it is depends on the rank of $W_{ij}^{(2)}$ and therefore also on the vanishing pattern of the 2-minors.

We are only left to consider the case when $T_{12}(y)$ is also of type I. The two factors of all three determinants agree if and only if

$$\widetilde{W} := \begin{pmatrix} F_{\bullet 0(0,1)} & w_{000}w_{111} + w_{010}w_{101} - w_{100}w_{011} - w_{110}w_{001} & F_{\bullet 1(0,1)} \\ F_{\bullet 0(0,2)} & w_{000}w_{112} + w_{010}w_{102} - w_{100}w_{012} - w_{110}w_{002} & F_{\bullet 1(0,2)} \\ F_{\bullet 0(1,2)} & w_{001}w_{112} + w_{011}w_{102} - w_{101}w_{012} - w_{111}w_{002} & F_{\bullet 1(1,2)} \end{pmatrix}$$

has rank one. One can compute that

$$V(\text{2-minors of } \widetilde{W}) = V(\text{3-minors of } W_I^{(3)}).$$

Hence, $\chi(V_I) = 2$ if and only if $\text{rank } W_I^{(3)} < 3$ and $\chi(V_I) = 1$ otherwise. \square

The last part of the above proof suggests that $\chi(Y_{W,2})$ does not depend solely on the vanishing pattern of the factors of E_A . Indeed, below we present a counterexample to Conjecture 1.3. This also shows that in general, the ML degree of independence models does not depend on the vanishing pattern of the factors of the principal A -determinant.

Example 3.16. Consider two scaling tensors $W, W' \in (\mathbb{C}^*)^{2 \times 2 \times 3}$ with

$$\begin{aligned}q_0 &= x_0y_0 + 3x_0y_1 + 2x_1y_0 + 4x_1y_1, \\ q_1 &= 2x_0y_0 + x_0y_1 + 4x_1y_0 + 6x_1y_1, \\ q_2 &= 3x_0y_0 + 4x_0y_1 + 6x_1y_0 + 10x_1y_1, \\ q'_2 &= 3x_0y_0 + 3x_0y_1 + 6x_1y_0 + x_1y_1,\end{aligned}$$

where q_0, q_1, q_2 are the quadrics associated to W and q_0, q_1, q'_2 are the quadrics associated to W' . For both we have $H_{012} = F_{0\bullet(0,1)} = F_{0\bullet(0,2)} = F_{0\bullet(1,2)} = 0$, but no other factor of E_A vanishes on W or W' . However, one can check that $\text{rank } W_{\{0,1,2\}}^{(3)} = 2$, while $\text{rank } W'_{\{0,1,2\}}^{(3)} = 3$.

Let us take a closer look at the corresponding systems $T(y) \cdot x = 0$. Denote

$$T(y) = \begin{pmatrix} y_0 + 3y_1 & 2y_2 + 4y_1 \\ 2y_0 + y_1 & 4y_0 + 6y_1 \\ 3y_0 + 4y_1 & 6y_0 + 10y_1 \end{pmatrix}, \quad T'(y) = \begin{pmatrix} y_0 + 3y_1 & 2y_0 + 4y_1 \\ 2y_0 + y_1 & 4y_0 + 6y_1 \\ 3y_0 + 3y_1 & 6y_0 + y_1 \end{pmatrix}$$

the matrices corresponding to W and W' respectively. Then

$$\begin{aligned} \det T_{01}(y) &= \det T'_{01}(y) = \det T_{02}(y) = 2y_1(4y_0 + 7y_1), \\ \det T_{02}(y) &= -2y_1(4y_0 + 7y_1), \\ \det T'_{02}(y) &= y_1(y_0 - 9y_1), \\ \det T'_{12}(y) &= -y_1(22y_0 + 17y_1) \end{aligned}$$

Indeed, $(T \begin{pmatrix} 7 \\ -4 \end{pmatrix}) \cdot \begin{pmatrix} 2 \\ -5 \end{pmatrix} = (T \begin{pmatrix} 1 \\ 0 \end{pmatrix}) \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix} = (T' \begin{pmatrix} 1 \\ 0 \end{pmatrix}) \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix} = 0$. The points $(\begin{pmatrix} 2 \\ -5 \end{pmatrix}, \begin{pmatrix} 7 \\ -4 \end{pmatrix})$ and $(\begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix})$ are solutions to $q_0 = q_1 = q_2 = 0$ and $(\begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix})$ is the single solution to $q_0 = q_1 = q'_2 = 0$. Hence, $\chi(V(q_0, q_1, q_2)) = 2$ and $\chi(V(q_0, q_1, q'_2)) = 1$. Overall, we obtain $\chi(Y_{W,2}) = -8$ and $\chi(Y_{W',2}) = -9$ and thus $\text{mldeg}(X_{W,n}) = 8$ and $\text{mldeg}(X_{W',n}) = 9$.

If we call the respective quadrics in $\mathbb{C}^1 \times \mathbb{C}^1$ as f_0, f_1, f_2, f'_2 , one can observe that $f_0 = f_1 = f_2 = 0$ has two solutions, one of them on the x -axis. In contrast, $f_0 = f_1 = f'_2 = 0$ has only one solution, which also lies on the x -axis. This comes from the fact that f_2 lies in the pencil of f_0 and f_1 while f'_2 does not, which again is directly linked to the ranks of the flattenings.

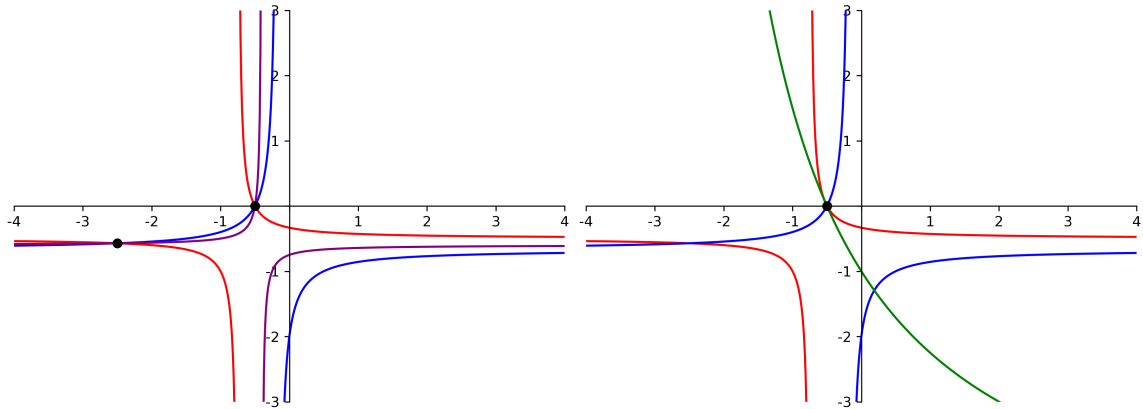


Figure 3: The red, blue, purple, and green quadrics are the vanishing loci of f_0, f_1, f_2 , and f'_2 , respectively.

4 Euler stratification for $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

In this section we present the complete Euler stratification for the hypersurface family $Y_{W,1}$. We start with a corollary to Theorem 3.1 relevant here.

Corollary 4.1. *Let $f_W(x_1, y_1, z_1, \dots, z_n)$ be the defining polynomial of $Y_{W,n} \subset (\mathbb{C}^*)^{2+n}$ and let $f_0, f_1, \dots, f_n \in \mathbb{C}[x_1^\pm, y_1^\pm]$ be as in (1). Then*

$$\chi(Y_{W,n}) = (-1)^n \chi(V(f_0 f_1 \cdots f_n)),$$

where $V(f_0 f_1 \cdots f_n) = \bigcup_{i=0}^n V(f_i)$ is the union of $n+1$ quadric curves in $(\mathbb{C}^*)^2$.

For $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ we have two quadrics Q_0 and Q_1 in $(\mathbb{C}^*)^2$ defined by

$$f_0 = w_{000} + w_{100}x + w_{010}y + w_{110}xy \quad \text{and} \quad f_1 = w_{001} + w_{101}x + w_{011}y + w_{111}xy.$$

Therefore, we need to characterize $\chi(Q_0 \cup Q_1)$ as $W \in (\mathbb{C}^*)^8$ varies.

Lemma 4.2. *Let $g(x, y) = a + bx + cy + dxy$ where $a, b, c, d \in \mathbb{C}^*$. Then the quadric $Q = V(g) \subset \mathbb{C}^2$ is nonsingular if and only if $ad - bc \neq 0$. In this situation, Q is a hyperbola and intersects the coordinate axes in two points, namely, in $(0, -a/c)$ and $(-a/b, 0)$. The Euler characteristic of Q in $(\mathbb{C}^*)^2$ is $\chi(Q) = -2$. If Q is singular, it is the union of the lines $x = -\frac{c}{d}$ and $y = -\frac{b}{d}$ that are parallel to the axes. In this case the Euler characteristic of Q in $(\mathbb{C}^*)^2$ is $\chi(Q) = -1$.*

Proof. Observe that

$$g(x, y) = \begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2}d & \frac{1}{2}b \\ \frac{1}{2}d & 0 & \frac{1}{2}c \\ \frac{1}{2}b & \frac{1}{2}c & a \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}.$$

Then, Q is nonsingular if and only if the middle matrix is nonsingular. The determinant of that matrix is $\frac{1}{4}d(bc - ad)$. If this determinant is zero, g factors as $g = d(x + \frac{c}{d})(y + \frac{b}{d})$. The computation of Euler characteristics follows easily. \square

Corollary 4.3. *Q_0 is nonsingular if and only if $F_{\bullet\bullet 0} = \det(W_{\bullet\bullet 0}) \neq 0$. Similarly, Q_1 is nonsingular if and only if $F_{\bullet\bullet 1} = \det(W_{\bullet\bullet 1}) \neq 0$.*

By Theorem 1.6, we will compute

$$\chi(Y_{W,1}) = -\chi(Q_0 \cup Q_1) = -(\chi(Q_0) + \chi(Q_1) - \chi(Q_0 \cap Q_1))$$

based only on the possible vanishing patterns of factors of E_A . We will determine these possible configurations throughout this section and we will study the effect of these vanishing patterns on $\chi(Q_0)$, $\chi(Q_1)$, and $\chi(Q_0 \cap Q_1)$. We use the same notation as in the subsection for $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ in Section 2.

4.1 Both Q_0 and Q_1 nonsingular

Here we are assuming that $F_{\bullet\bullet 0}$ and $F_{\bullet\bullet 1}$, the determinants of the two slices $W_{\bullet\bullet 0}$ and $W_{\bullet\bullet 1}$, are not zero. This means that $\chi(Q_0) = \chi(Q_1) = -2$, and we need to determine $\chi(Q_0 \cap Q_1)$. Hence we need to determine all the possible ways Q_0 and Q_1 intersect in $(\mathbb{C}^*)^2$.

Consider

$$w_{111}f_0 - w_{110}f_1 = \det(W_{1\bullet\bullet})x + \det(W_{\bullet 1\bullet})y + (w_{000}w_{111} - w_{110}w_{001}). \quad (5)$$

If $F_{\bullet 1 \bullet} \neq 0$, we can solve for y in (5). Plugging this into f_0 gives

$$G_1(x) = \det(W_{1 \bullet \bullet})x^2 + [(w_{000}w_{111} - w_{110}w_{001}) - (w_{010}w_{101} - w_{100}w_{011})]x + \det(W_{0 \bullet \bullet}) = 0.$$

Similarly, if $F_{1 \bullet \bullet} \neq 0$, we can solve for x in (5). Plugging this into f_0 gives

$$G_2(y) = \det(W_{\bullet 1 \bullet})y^2 + [(w_{000}w_{111} - w_{110}w_{001}) + (w_{010}w_{101} - w_{100}w_{011})]y + \det(W_{\bullet 0 \bullet}) = 0.$$

The discriminants of both $G_1(x)$ and $G_2(y)$ are equal to the hyperdeterminant H .

Lemma 4.4. *The two nonsingular quadrics Q_0 and Q_1 intersect in two distinct points in \mathbb{C}^2 if $F_{\bullet 1 \bullet} \neq 0$, $F_{1 \bullet \bullet} \neq 0$, and $H \neq 0$. In this case, they intersect in a point where $x = 0$ if and only if $F_{0 \bullet \bullet} = 0$. Similarly, they intersect in a point where $y = 0$ if and only if $F_{\bullet 0 \bullet} = 0$.*

Proof. If the three polynomials do not vanish, then $G_1(x)$ (and $G_2(y)$) has two distinct roots x_1 and x_2 , and one can solve for the y coordinate of the two intersection points using (5). Under these conditions, $G_1(x) = 0$ implies that one its roots will be 0 if and only if $\det(W_{0 \bullet \bullet}) = F_{0 \bullet \bullet} = 0$. Similarly, $G_2(y) = 0$ implies that one its roots will be 0 if and only if $\det(W_{\bullet 0 \bullet}) = F_{\bullet 0 \bullet} = 0$. \square

$F_{\bullet 1 \bullet} \neq 0$, $F_{1 \bullet \bullet} \neq 0$, and $H \neq 0$

There are only two factors left to consider and the following captures the implications of their vanishings.

Proposition 4.5. *Suppose Q_0 and Q_1 are nonsingular quadrics and $F_{\bullet 1 \bullet} \neq 0$, $F_{1 \bullet \bullet} \neq 0$, and $H \neq 0$.*

- *If neither $F_{0 \bullet \bullet}$ nor $F_{\bullet 0 \bullet}$ vanishes, then $\chi(Y_{W,1}) = 6$.*
- *If exactly one of $F_{0 \bullet \bullet}$ and $F_{\bullet 0 \bullet}$ vanishes, then $\chi(Y_{W,1}) = 5$.*
- *If both $F_{0 \bullet \bullet}$ and $F_{\bullet 0 \bullet}$ vanish, then $\chi(Y_{W,1}) = 4$.*

Proof. By Lemma 4.4, the quadrics intersect in exactly two points in $(\mathbb{C}^*)^2$ if the constant terms of $G_1(x)$ and $G_2(y)$ do not vanish. In this case, $\chi(Q_0 \cap Q_1) = 2$ and hence $\chi(Q_0 \cup Q_1) = -6$. If exactly one of these constant terms vanish, exactly one of these points is on a coordinate axis. This means $\chi(Q_0 \cap Q_1) = 1$ and hence $\chi(Q_0 \cup Q_1) = -5$. If both constant terms are zero, both intersection points are on the coordinate axes. Therefore, $\chi(Q_0 \cap Q_1) = 0$ and $\chi(Q_0 \cup Q_1) = -4$. \square

We point out that the second case above, where $\chi(Y_{W,1}) = 5$, corresponds to exactly one of the minors of W vanishing. This is the case A in Figure 2. The third case, where $\chi(Y_{W,1}) = 4$, corresponds to exactly two minors which form a hook vanishing. This is the case C in Figure 2.

$F_{\bullet 1\bullet} \neq 0, F_{1\bullet\bullet} \neq 0,$ and $H = 0$

The vanishing of the hyperdeterminant implies that the quadrics Q_0 and Q_1 are tangent to each other.

Lemma 4.6. *Suppose the quadrics Q_0 and Q_1 are nonsingular and $F_{\bullet 1\bullet} \neq 0$ and $F_{1\bullet\bullet} \neq 0$. If the hyperdeterminant vanishes, Q_0 and Q_1 intersect in \mathbb{C}^2 in a single point of multiplicity two. Furthermore,*

1. *If neither $F_{0\bullet\bullet}$ nor $F_{\bullet 0\bullet}$ vanishes, then the unique intersection point is in $(\mathbb{C}^*)^2$.*
2. *If exactly one of $F_{0\bullet\bullet}$ and $F_{\bullet 0\bullet}$ vanishes, then the unique intersection point is on one of the coordinate axes.*

Proof. Under the hypotheses, $G_1(x)$ and $G_2(y)$ have a double root. This means that Q_0 and Q_1 intersect in a single point of tangency. If the constant terms of $G_1(x)$ and $G_2(y)$, namely, $F_{0\bullet\bullet}$ and $F_{\bullet 0\bullet}$, do not vanish, this intersection point is in $(\mathbb{C}^*)^2$. If exactly one of them vanishes, then $x = 0$ is a double root of $G_1(x)$ or $y = 0$ is a double root of $G_2(y)$. \square

The alert reader would realize that we have omitted the case when $F_{0\bullet\bullet} = F_{\bullet 0\bullet} = 0$. This corresponds to the case when the minors in a hook and the hyperdeterminant vanish. However, this forces the minors in the cubic frame containing the hook all vanish. The extra minors are $F_{\bullet 1\bullet}$ and $F_{1\bullet\bullet}$ which we assumed to be nonzero.

Proposition 4.7. *Suppose Q_0 and Q_1 are nonsingular quadrics and $F_{\bullet 1\bullet} \neq 0, F_{1\bullet\bullet} \neq 0,$ and $H = 0$.*

- *If neither $F_{0\bullet\bullet}$ nor $F_{\bullet 0\bullet}$ vanishes, then $\chi(Y_{W,1}) = 5$.*
- *If exactly one of $F_{0\bullet\bullet}$ and $F_{\bullet 0\bullet}$ vanishes, then $\chi(Y_{W,1}) = 4$.*

Proof. By Lemma 4.6, the intersection $Q_0 \cap Q_1$ consists of a single point. In the first case, this point is in $(\mathbb{C}^*)^2$. Therefore $\chi(Q_0 \cap Q_1) = 1$ and $\chi(Y_{W,1}) = 5$. In the second case $Q_0 \cap Q_1 = \emptyset$ in $(\mathbb{C}^*)^2$. Therefore $\chi(Q_0 \cap Q_1) = 0$ and $\chi(Y_{W,1}) = 4$. \square

The first case above, where $\chi(Y_{W,1}) = 5$, corresponds to exactly the hyperdeterminant vanishing. This is the case B in Figure 2. The second case where $\chi(Y_{W,1}) = 4$ corresponds to one minor plus the hyperdeterminant vanishing. This is the case E in Figure 2.

Exactly one of $F_{\bullet 1\bullet}$ and $F_{1\bullet\bullet}$ vanishes, and $H \neq 0$

Without loss of generality, we will assume $F_{\bullet 1\bullet} \neq 0$ but $F_{1\bullet\bullet} = 0$ since the other case leads to similar conclusions by symmetry (via exchanging the roles of the variables x and y).

Lemma 4.8. *Suppose the quadrics Q_0 and Q_1 are nonsingular, and $F_{\bullet 1\bullet} \neq 0, F_{1\bullet\bullet} = 0,$ and $H \neq 0$. Then Q_0 and Q_1 intersect in a unique point in \mathbb{C}^2 . Furthermore,*

1. *If neither $F_{0\bullet\bullet}$ nor $F_{\bullet 0\bullet}$ vanishes, then this unique intersection point is in $(\mathbb{C}^*)^2$.*

2. If exactly one of $F_{0\bullet\bullet}$ and $F_{\bullet 0\bullet}$ vanishes, then the unique intersection point is on one of the coordinate axes.

Proof. When only $F_{1\bullet\bullet}$ vanishes, then in (5) we can uniquely solve for y . In addition, $H \neq 0$ implies that $G_1(x)$ has a unique root. Therefore Q_0 and Q_1 intersect in a unique point in \mathbb{C}^2 . If $F_{0\bullet\bullet}$ vanishes, then $x = 0$ is the unique root of $G_1(x)$. If $F_{\bullet 0\bullet}$ vanishes, we see that $w_{000}w_{111} - w_{110}w_{001}$ is in $\langle F_{1\bullet\bullet}, F_{\bullet 0\bullet} \rangle : (\prod w_{ijk})^\infty$. This means that the constant term in (5) is zero. Therefore $y = 0$. \square

Again, we have not considered the case $F_{0\bullet\bullet} = F_{\bullet 0\bullet} = 0$, since together with $F_{1\bullet\bullet} = 0$ we have a square cup with all minors vanishing. However, this implies that the missing minor in the cubic frame containing this square cup, namely, $F_{\bullet 1\bullet}$, as well as H vanish. However, we assumed that these polynomials are not zero.

Proposition 4.9. *Suppose Q_0 and Q_1 are nonsingular quadrics, and $F_{\bullet 1\bullet} \neq 0$, $F_{1\bullet\bullet} = 0$, and $H \neq 0$.*

- If neither $F_{0\bullet\bullet}$ nor $F_{\bullet 0\bullet}$ vanishes, then $\chi(Y_{W,1}) = 5$.
- If exactly one of $F_{0\bullet\bullet}$ and $F_{\bullet 0\bullet}$ vanishes, then $\chi(Y_{W,1}) = 4$.

Proof. The proof is identical to the proof of Proposition 4.7. \square

The first case above, where $\chi(Y_{W,1}) = 5$, corresponds to exactly one minor of W vanishing. This is the case A in Figure 2. The second case, where $\chi(Y_{W,1}) = 4$, corresponds to either the minors in a hook or in a mirror vanishing. These are the cases C and D in Figure 2, respectively.

Exactly one of $F_{\bullet 1\bullet}$ and $F_{1\bullet\bullet}$ vanishes, and $H = 0$

Again by symmetry we assume that $F_{\bullet 1\bullet} \neq 0$ and $F_{1\bullet\bullet} = H = 0$. We immediately see that $G_1(x) = F_{0\bullet\bullet}$. Therefore, if $F_{0\bullet\bullet} \neq 0$, the quadrics Q_0 and Q_1 do not intersect. Otherwise, the minors in a mirror and H vanish. This implies that the minors in a cubic frame containing the mirror also vanish. But this contradicts our assumption that Q_0 and Q_1 are nonsingular, as well as $F_{\bullet 1\bullet} \neq 0$. As a result, we conclude the following.

Proposition 4.10. *Suppose the quadrics Q_0 and Q_1 are nonsingular, exactly one of $F_{\bullet 1\bullet}$ and $F_{1\bullet\bullet}$ vanishes, and $H = 0$ and the remaining minors of W do not vanish. Then Q_0 and Q_1 do not intersect in \mathbb{C}^2 and $\chi(Y_{W,1}) = 4$.*

The only case above, where $\chi(Y_{W,1}) = 4$, corresponds to exactly one minor of W and the hyperdeterminant vanishing. This is the case E in Figure 2.

$F_{\bullet 1 \bullet} = F_{1 \bullet \bullet} = 0$ and $H \neq 0$

Lemma 4.11. *Suppose the quadrics Q_0 and Q_1 are nonsingular. If $F_{\bullet 1 \bullet} = F_{1 \bullet \bullet} = 0$ and $H \neq 0$, then Q_0 and Q_1 do not intersect in \mathbb{C}^2 if and only if $F_{0 \bullet \bullet} \neq 0$.*

Proof. Under the above hypotheses, the quadrics do not intersect if and only if the constant term $w_{000}w_{111} - w_{110}w_{001}$ in (5) is not equal to zero. This constant term is the determinant of the first and third columns of the matrix $\begin{bmatrix} w_{000} & w_{010} & w_{110} \\ w_{001} & w_{001} & w_{111} \end{bmatrix}$. The determinant of the last two columns is $F_{\bullet 1 \bullet} = 0$. Since the determinant of the first two columns is $F_{0 \bullet \bullet}$, we conclude that $w_{000}w_{111} - w_{110}w_{001} \neq 0$ if and only if $F_{0 \bullet \bullet} \neq 0$. \square

Proposition 4.12. *Suppose the quadrics Q_0 and Q_1 are nonsingular, both $F_{\bullet 1 \bullet}$ and $F_{1 \bullet \bullet}$ vanish, but H , $F_{0 \bullet \bullet}$ and $F_{\bullet 0 \bullet}$ do not vanish. Then Q_0 and Q_1 do not intersect in \mathbb{C}^2 and $\chi(Y_{W,1}) = 4$.*

The case in the above proposition, where $\chi(Y_{W,1}) = 4$, has the minors in a hook vanishing. This is the case C in Figure 2. Note that by Lemma 4.11, if $F_{0 \bullet \bullet} = 0$ the the quadrics intersect. However, the three vanishing minors form a square cup, and this implies the vanishing of all the minors in the cubic frame containing the square cup and the hyperdeterminant. Since we assumed $H \neq 0$, we do not consider this case here.

$F_{\bullet 1 \bullet} = F_{1 \bullet \bullet} = H = 0$

Proposition 4.13. *Suppose the quadrics Q_0 and Q_1 are nonsingular and $F_{\bullet 1 \bullet} = F_{1 \bullet \bullet} = H = 0$. This implies that $F_{0 \bullet \bullet} = F_{\bullet 0 \bullet} = 0$. In this case, $Q_0 = Q_1$ and $\chi(Y_{W,1}) = 2$.*

Proof. It is easy to compute that the radical of the ideal $\langle H, F_{\bullet 1 \bullet}, F_{1 \bullet \bullet} \rangle : (\prod w_{ijk})^\infty$ contains $F_{0 \bullet \bullet}$ and $F_{\bullet 0 \bullet}$. This implies that the coefficients of f_1 is a constant multiple of the coefficients of f_0 . Hence, $Q_0 = Q_1$ and the result follows. \square

This last case corresponds to the minors in a cubic frame and the hyperdeterminant vanishing. This is the case G in Figure 2.

4.2 Q_0 nonsingular and Q_1 singular

Now we assume that $F_{\bullet \bullet 0} \neq 0$ and $F_{\bullet \bullet 1} = 0$. In other words, Q_0 is nonsingular but Q_1 is the union of the lines L_1 and L_2 defined by $x = -\frac{w_{011}}{w_{111}}$ and $y = -\frac{w_{101}}{w_{111}}$, respectively. This means that $\chi(Q_0) = -2$ and $\chi(Q_1) = -1$. Again, we need to determine all the possible ways Q_0 and Q_1 intersect in $(\mathbb{C}^*)^2$.

$F_{\bullet 1 \bullet} \neq 0$, $F_{1 \bullet \bullet} \neq 0$, and $H \neq 0$

We start out with pointing out that Lemma 4.4 remains valid even if we relax the condition on the nonsingularity of Q_1 . Therefore we get the analog of Proposition 4.5.

Proposition 4.14. *Suppose Q_0 is a nonsingular quadric and Q_1 is a singular quadric. Moreover, suppose that $F_{\bullet 1 \bullet} \neq 0$, $F_{1 \bullet \bullet} \neq 0$, and $H \neq 0$.*

- If neither $F_{0\bullet\bullet}$ nor $F_{\bullet 0\bullet}$ vanishes, then $\chi(Y_{W,1}) = 5$.
- If exactly one of $F_{0\bullet\bullet}$ and $F_{\bullet 0\bullet}$ vanishes, then $\chi(Y_{W,1}) = 4$.
- If both $F_{0\bullet\bullet}$ and $F_{\bullet 0\bullet}$ vanish, then $\chi(Y_{W,1}) = 3$.

Proof. The proof is identical to the proof of Proposition 4.5, except now $\chi(Q_0) + \chi(Q_1) = -3$. \square

We again point out that the first case above, where $\chi(Y_{W,1}) = 5$, corresponds to exactly one of the minors of W vanishing. This is the case A in Figure 2. The second case where $\chi(Y_{W,1}) = 4$ corresponds to exactly two minors which form a hook vanishing. This is the case C in Figure 2. Finally, the third case where $\chi(Y_{W,1}) = 3$ corresponds to three minors meeting at a vertex of W vanishing. This is the case F.

$F_{\bullet 1\bullet} \neq 0, F_{1\bullet\bullet} \neq 0, \text{ and } H = 0$

Now Lemma 4.6 stays valid except its last statement. The two quadrics intersect in a single point of multiplicity two: the intersection point of L_1 and L_2 lies on Q_0 . Moreover, if neither $F_{0\bullet\bullet}$ nor $F_{\bullet 0\bullet}$ vanishes, this intersection point is in $(\mathbb{C}^*)^2$. If at least one of these vanishes, we will have the minors of a hook and hyperdeterminant vanish. But then all minors in the cubic frame containing this hook, including $F_{\bullet\bullet 0}$, are equal to zero. But we assumed that Q_0 is nonsingular.

Proposition 4.15. *Suppose Q_0 is a nonsingular quadric and Q_1 is a singular quadric. If only the hyperdeterminant H vanishes, then $\chi(Y_{W,1}) = 4$.*

Proof. Based on the observations above we conclude that $\chi(Q_0 \cap Q_1) = -1$. \square

This case, where $\chi(Y_{W,1}) = 4$, corresponds to exactly one minor of W and the hyperdeterminant H vanishing. This is the case E in Figure 2.

Exactly one of $F_{\bullet 1\bullet}$ and $F_{1\bullet\bullet}$ vanishes, and $H \neq 0$

Without loss of generality, we will assume $F_{\bullet 1\bullet} \neq 0$ but $F_{1\bullet\bullet} = 0$. This time, Lemma 4.8 stays valid with a small modification to its last statement. As before, Q_0 and Q_1 intersect in a unique point. This point is the intersection of Q_0 and L_1 , and L_2 does not intersect Q_0 at all. The intersection point is in $(\mathbb{C}^*)^2$ if neither $F_{0\bullet\bullet}$ nor $F_{\bullet 0\bullet}$ vanishes. Note that if $F_{0\bullet\bullet}$ vanishes we have the three minors of a square cup vanishing. And this forces $F_{\bullet\bullet 0} = 0$, implying that Q_0 is singular. But we are assuming Q_0 is nonsingular. On the other hand, if $F_{\bullet 0\bullet}$ vanishes, Q_0 and Q_1 intersect in a unique point on $y = 0$. Hence, we get the following result.

Proposition 4.16. *Suppose Q_0 is a nonsingular quadric and Q_1 is a singular quadric. Moreover, suppose that $F_{\bullet 1\bullet} \neq 0, F_{1\bullet\bullet} = 0, \text{ and } H \neq 0$.*

- If neither $F_{0\bullet\bullet}$ nor $F_{\bullet 0\bullet}$ vanishes, then $\chi(Y_{W,1}) = 4$.

- If $F_{\bullet 0 \bullet}$ vanishes, then $\chi(Y_{W,1}) = 3$.

Proof. The first case corresponds to the configuration where Q_0 and Q_1 intersect in a unique point in $(\mathbb{C}^*)^2$. Therefore $\chi(Q_0 \cap Q_1) = -1$. In the second case, this unique intersection point is on $y = 0$, and hence $\chi(Q_0 \cap Q_1) = 0$. \square

The first case, where $\chi(Y_{W,1}) = 4$, corresponds to the minors of a hook vanishing. This is the case C in Figure 2. The second case, where $\chi(Y_{W,1}) = 3$, corresponds to three minors touching a vertex of W vanishing. This is the case F.

The next possibility is where exactly one of $F_{\bullet 1 \bullet}$ and $F_{1 \bullet \bullet}$ as well as H vanish. However, both lead to a configuration where the minors of a hook and the hyperdeterminant vanish. This forces the minors of the cubic frame containing the hook to vanish. However, one of these minors is $F_{\bullet \bullet 0}$, and therefore Q_0 would be singular; a contradiction to our assumption. So we skip this case.

$F_{\bullet 1 \bullet} = F_{1 \bullet \bullet} = 0$ and $H \neq 0$

As in Lemma 4.11, if the remaining two minors do not vanish, then Q_0 and Q_1 do not intersect in \mathbb{C}^2 . This is equivalent to L_1 and L_2 being the asymptotes of Q_0 .

Proposition 4.17. *Suppose the quadric Q_0 is nonsingular and Q_1 is singular. Further suppose that both $F_{\bullet 1 \bullet}$ and $F_{1 \bullet \bullet}$ vanish, but H , $F_{0 \bullet \bullet}$ and $F_{\bullet 0 \bullet}$ do not vanish. Then Q_0 and Q_1 do not intersect in \mathbb{C}^2 and $\chi(Y_{W,1}) = 3$.*

The above case, where $\chi(Y_{W,1}) = 3$, corresponds to three minors touching a common vertex of W vanishing. This is the case F in Figure 2.

Finally, we see that we need to skip the possibility where $F_{\bullet 1 \bullet} = F_{1 \bullet \bullet} = H = 0$, since as before, this forces Q_0 to be singular.

4.3 Both Q_0 and Q_1 singular

In the last few cases, we need to consider both Q_0 and Q_1 as unions of lines: $Q_0 = K_1 \cup K_2 = \{x = -\frac{w_{010}}{w_{110}}\} \cup \{y = -\frac{w_{100}}{w_{110}}\}$ and $Q_1 = L_1 \cup L_2 = \{x = -\frac{w_{011}}{w_{111}}\} \cup \{y = -\frac{w_{101}}{w_{111}}\}$. Based on our work in Section 2 and our discussion so far in this section, we note that as soon as a single extra factor of the principal A -determinant E_A vanishes, then all minors of a cubic frame containing the minors $F_{\bullet \bullet 0}$ and $F_{\bullet \bullet 1}$, as well as the hyperdeterminant H vanish. If yet another one of the remaining two factors vanish, then all factors of E_A vanish. Therefore, we just have to deal with these three possibilities. We summarize the corresponding result in the following.

Proposition 4.18. *Suppose both quadrics Q_0 and Q_1 are singular.*

- If no other factor of E_A vanishes, then $\chi(Y_{W,1}) = 4$.
- If $F_{0 \bullet \bullet} = F_{1 \bullet \bullet} = H = 0$ or $F_{\bullet 0 \bullet} = F_{\bullet 1 \bullet} = H = 0$ and no other factor vanishes, then $\chi(Y_{W,1}) = 2$.

- If all factors of E_A vanish, then $\chi(Y_{W,1}) = 1$.

Proof. In the first case, Lemma 4.4 implies that Q_0 and Q_1 intersect in two points in $(\mathbb{C}^*)^2$. Since $\chi(Q_0) = \chi(Q_1) = -1$, we conclude that $\chi(Y_{W,1}) = 4$. The second case corresponds to the situation where $K_1 = L_1$ and $K_2 \neq L_2$ or $K_1 \neq L_1$ and $K_2 = L_2$. Since the Euler characteristic of a line of the form $x = a \neq 0$ or $y = b \neq 0$ in $(\mathbb{C}^*)^2$ is 0, we get $\chi(Y_{W,1}) = 2$. Finally, if all factors of E_A vanish, this forces all coefficients of f_1 to be a constant multiple of the corresponding coefficients of f_0 . Therefore, $Q_0 = Q_1$, and $\chi(Y_{W,1}) = 1$. \square

Now we put together all the results above in the proof of Theorem 1.7.

Proof of Theorem 1.7: When no factor of E_A vanishes the ML degree is maximum possible and is equal to $\deg(\mathcal{X}_1) = 6$. By the second case of Proposition 4.5, the first case of Proposition 4.7, the first case of Proposition 4.9, and the first case of Proposition 4.14 together with the symmetries of our $2 \times 2 \times 2$ tensor W , we see that when exactly a single factor of E_A vanishes, then $\chi(Y_{W,1}) = 5$. By the last case of Proposition 4.5, the second case of Proposition 4.7, the second case of Proposition 4.9, Proposition 4.10, Proposition 4.12, the second case of Proposition 4.14, Proposition 4.15, the first case of Proposition 4.16, and the first case Proposition 4.18 together with the symmetries of W , we conclude that the two minors in every mirror, every hook, and any minor together with the hyperdeterminant can vanish on their own. This accounts for all $\binom{7}{2}$ pairs of factors of E_A . In all of these cases $\chi(Y_{W,1}) = 4$. The only possibilities when exactly three factors vanish appear in the last case of Proposition 4.14, the second case of Proposition 4.16, and Proposition 4.17. These are cases when three 2-minors meeting at a corner of the tensor W vanish. By symmetry there are eight of them and the Euler characteristic is $\chi(Y_{W,1}) = 3$. We do not find a possibility where a subset of four factors of E_A vanish. The only time five factors can vanish simultaneously is given by Proposition 4.13 and the second case of Proposition 4.18. These correspond the 2-minors in a cubic frame and the hyperdeterminant vanishing. By symmetry there are three such cases and then $\chi(Y_{W,1}) = 2$. We also do not find a possibility where all but one factor of E_A simultaneously vanish. Finally, when all factors vanish we see that $\chi(Y_{W,1}) = 1$. \square

5 Realizability of ML degrees

In this section we give a positive answer to the question of the realizability of $\text{mldeg}(X_{W,n})$ by proving Theorem 1.8. Throughout this section, Q_k will denote the quadric curve in $(\mathbb{C}^*)^2$ defined by $f_k = w_{00k} + w_{10k}x + w_{01k}y + w_{11k}xy$ for $k = 0, \dots, n$ as in (1). In order to prove the result, we construct a set of scalings $W \in (\mathbb{C}^*)^{4n+4}$ whose corresponding arrangement of quadric curves achieve every possible Euler characteristic. We start with the following simple corollary.

Corollary 5.1. *Let $W \in (\mathbb{C}^*)^{4n+4}$ be such that W does not cause any $2 \times 2 \times 3$ hyperdeterminantal factors of E_A to vanish, then we have the following equality:*

$$\text{mldeg}(X_{W,n}) = (-1)^{n+1} \chi(Y_{W,n}) = - \left(\sum_{k=0}^n \chi(Q_k) - \sum_{\substack{j,k=0 \\ j \neq k}}^n |Q_j \cap Q_k| \right)$$

Proof. By Corollary 4.1, we need to compute $\chi(Q_0 \cup Q_1 \cup \dots \cup Q_n)$. Using inclusion/exclusion and the fact that a triple intersection $Q_i \cap Q_j \cap Q_k$ in \mathbb{C}^2 is nonempty if and only if the corresponding hyperdeterminant H_{ijk} vanishes proves the result. \square

We recall that, for $0 \leq j \neq k \leq n$, a choice of W such that $F_{\bullet 1(j,k)} = F_{0 \bullet(j,k)} = 0$ and no other relations among their coefficients are satisfied gives us two nonsingular quadrics with $Q_j \cap Q_k = \emptyset$ in the torus. It follows from Lemma 4.8 that Q_j and Q_k intersect in a unique point in \mathbb{C}^2 and this point is on $y = 0$. Likewise, if we pick W such that $F_{\bullet 0(j,k)} = F_{1 \bullet(j,k)} = 0$, then the intersection point is on $x = 0$. Figure 4 illustrates this situation.

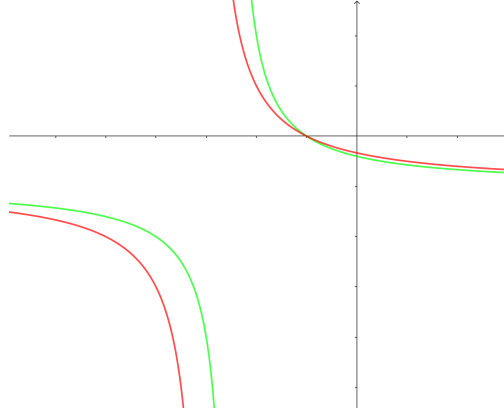


Figure 4: Two nonsingular quadrics Q_j, Q_k where $F_{\bullet 1(j,k)} = F_{0 \bullet(j,k)} = 0$

Definition 5.2. The set $\text{altH}(n)$ of alternating hook minors consists of the factors of E_A given by $F_{\bullet 1(a,a+1)}, F_{0 \bullet(a,a+1)}, F_{\bullet 0(b,b+1)}, F_{1 \bullet(b,b+1)}$ as a ranges over all odd numbers in $[n]$ and b ranges over all even numbers in $[\overline{n}]$. See Figure 5.

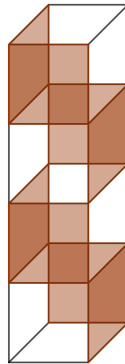


Figure 5: The alternating hook minors $\text{altH}(4)$

The alternating hook minors of E_A enjoy the following nice property.

Proposition 5.3. *For any subset $S \subset \text{altH}(n) \cup \{F_{\bullet\bullet n}\}$, if $W \in (\mathbb{C}^*)^{4n+4}$ is chosen to be a generic solution to the system given by S , then no other factor of E_A will vanish at W .*

Proof. Since S only contains 2×2 determinant factors of E_A and it contains no cups as well as no overlapping minors on a face of W , no other 2×2 determinant factors of E_A vanishes. Furthermore, since for any choice of distinct $i, j \in [\bar{n}]$, the subtensor given by the coefficients of f_i, f_j makes at most a pair of minors in that subtensor vanish, a generic W cannot cause any $2 \times 2 \times 2$ hyperdeterminantal factor of E_A to vanish. Lastly, for any pairwise distinct $i, j, k \in [\bar{n}]$, by Lemma 2.11, we do not have enough minors in the corresponding $2 \times 2 \times 3$ subtensor vanish that would cause the hyperdeterminant H_{ijk} to vanish. \square

A consequence of the above result is that we can use Corollary 5.1 to compute $\text{mldeg}(X_{W,n})$ when W satisfies the conditions stipulated by Proposition 5.3.

Lemma 5.4. *For any subset $S \subset \text{altH}(n) \cup \{F_{\bullet\bullet n}\}$, if $W \in (\mathbb{C}^*)^{4n+4}$ is chosen to be a generic solution to the system given by S , then $\text{mldeg}(X_{W,n}) = (n+2)(n+1) - |S|$.*

Proof. We prove this statement by directly counting intersection points and summing Euler characteristics according to Corollary 5.1. If S contains $F_{\bullet\bullet n}$, then $\sum_{k=0}^n \chi(Q_k) = -(2n+1)$, since exactly one quadric is singular. Meanwhile, for every 2-minor in S , the pair of quadrics whose coefficients overlap with those of this minor will either have one of their two intersection points on an axis or share an asymptote, meaning that the number of pairwise intersection points among the quadrics will be $((n+1)n - (|S| - 1))$. Corollary 5.1 implies that $\text{mldeg}(X_{W,n}) = (n+2)(n+1) - |S|$. If S does not contain $F_{\bullet\bullet n}$, then $\sum_{k=0}^n \chi(Q_k) = -(2n+2)$. By the same argument as for the first case, the number of pairwise intersection points among the quadrics will be $((n+1)n - |S|)$. Therefore $\text{mldeg}(X_{W,n}) = (n+2)(n+1) - |S|$. \square

Example 5.5. We give an example for $n = 2$ where we pick $S = \text{altH}(2) \cup \{F_{\bullet\bullet 2}\}$. Note that $|S| = 5$. We use W with quadrics

$$\begin{aligned} f_0 &= 2 + x + 5y + 2xy, \\ f_1 &= 2 + 2x + 5y + 2xy, \\ f_2 &= 1 + x + y + xy. \end{aligned}$$

We see that Q_0 and Q_1 are nonsingular while Q_2 is singular. Therefore, their Euler characteristics are $-2, -2, -1$ respectively. Figure 6 shows the arrangement of these three quadrics with two pairwise intersection points in the torus.

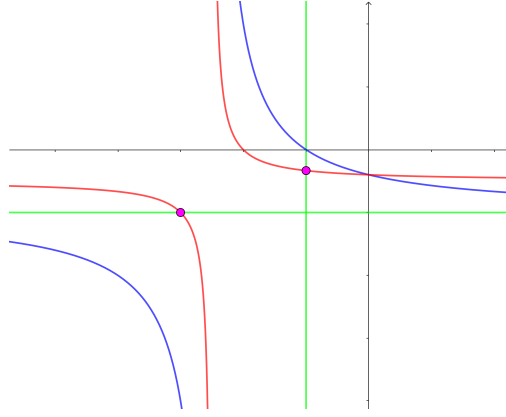


Figure 6: Three quadratics with $S = \text{altH}(2) \cup F_{\bullet\bullet 2}$.

We conclude that $\text{mldeg}(X_{W,2}) = |-2 - 2 - 1 - 2| = 7$, which is equal to $4 \cdot 3 - 5$.

Definition 5.6. The feasible set $\mathcal{A}_n \subset [(n+2)(n+1)]$ is the subset with the property that for every $\alpha \in \mathcal{A}_n$, there exists some $W \in (\mathbb{C}^*)^{4n+4}$ such that $\text{mldeg}(X_{W,n}) = \alpha$.

Lemma 5.7. For all $n \geq 2$, $\mathcal{A}_{n-1} \subset \mathcal{A}_n$

Proof. Consider the set of all W such that $W_{\bullet\bullet(n-1)} = W_{\bullet\bullet n}$. Then for any choice of W in this set, $Q_n = Q_{n-1}$ and therefore

$$\chi(Y_{W,n}) = (-1)^n \chi(V(f_0 f_1 \cdots f_n) \cap (\mathbb{C}^*)^2) = (-1)^n \chi(V(f_0 f_1 \cdots f_{n-1}) \cap (\mathbb{C}^*)^2).$$

□

Proof of Theorem 1.8: We use induction on n , where the base case of $n = 1$ is covered by Theorem 1.7. By Lemma 5.7, it suffices to show that $\{(n+1)n+1, \dots, (n+2)(n+1)\} \subset \mathcal{A}_n$. For any $\alpha \in \{(n+1)n+1, \dots, (n+2)(n+1)\}$, we pick $S \subset \text{altH}(n) \cup \{F_{\bullet\bullet n}\}$ so that $|S| = (n+2)(n+1) - \alpha$ and a scaling $W \in (\mathbb{C}^*)^{4n+4}$ which is a generic solution to the system given by S . Then by Lemma 5.4, $\text{mldeg}(X_{W,n}) = (n+2)(n+1) - ((n+2)(n+1) - \alpha) = \alpha$. □

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References

- [ABB⁺19] Carlos Améndola, Nathan Bliss, Isaac Burke, Courtney R. Gibbons, Martin Helmer, Serkan Hoşten, Evan D. Nash, Jose Israel Rodriguez, and Daniel Smolkin. The maximum likelihood degree of toric varieties. *Journal of Symbolic Computation*, 92:222–242, 2019.
- [CHKO24] Oliver Clarke, Serkan Hoşten, Nataliia Kushnerchuk, and Janike Oldekop. Matroid stratification of ML degrees of independence models. *Algebraic Statistics*, 15(2):199–223, 2024.
- [CHKS06] Fabrizio Catanese, Serkan Hoşten, Amit Khetan, and Bernd Sturmfels. The maximum likelihood degree. *American Journal of Mathematics*, 128(3):671–697, 2006.
- [FMH24] Claudia Fevola and Saiei-Jaeyeong Matsubara-Heo. Euler discriminant of complements of hyperplanes, 2024. [arXiv:2407.18176](https://arxiv.org/abs/2407.18176).
- [GKZ94] Izrail M. Gelfand, Mikhail M. Kapranov, and Andrei V. Zelevinsky. *Discriminants, resultants, and multidimensional determinants*. Mathematics: Theory & Applications, Birkhäuser Boston, Inc., Boston, MA, 1994.
- [HKS05] Serkan Hoşten, Amit Khetan, and Bernd Sturmfels. Solving the likelihood equations. *Foundations of Computational Mathematics*, 5(4):389–407, 2005.
- [HS14] June Huh and Bernd Sturmfels. *Likelihood Geometry*, pages 63–117. Lecture Notes in Mathematics. Springer International Publishing, 2014.
- [Huh13] June Huh. The maximum likelihood degree of a very affine variety. *Compositio Mathematica*, 149(8):1245–1266, 2013.
- [Sul18] Seth Sullivant. *Algebraic Statistics*. Graduate Studies in Mathematics. American Mathematical Society, 2018.
- [TW24] Simon Telen and Maximilian Wiesmann. Euler stratifications of hypersurface families, 2024. [arXiv:2407.18176](https://arxiv.org/abs/2407.18176).
- [WZ96] Jerzy Weyman and Andrei Zelevinsky. Singularities of hyperdeterminants. *Annales de l’Institut Fourier*, 46(3):591–644, 1996.

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